An explicit construction of K3 surfaces of genus 11

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Overview

- Unirational parametrization
- Polarized $K3$ surfaces and their moduli spaces
- Mukai's unirationality constructions for small genus
- Polarized $K3$ surfaces of genus 11

(Uni-)rational parametrization

Definition

An algebraic variety X is **unirational** if there exits a dominant rational map

$$\phi : \mathbb{P}^n \dashrightarrow X.$$

The variety X is **rational** if ϕ is birational.

Circle

A rational parametrization can be achieved by a **stereographic projection**

$$\mathbb{A}^1(\Bbbk) \dashrightarrow \mathbb{A}^2(\Bbbk), t \mapsto \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$$

in the affine chart or projectively $\mathbb{P}^1 \to \mathbb{P}^2, (s:t) \mapsto (2st : t^2 - s^2 : s^2 + t^2)$.

Example: Grassmannian $\mathbb{G}(2,4)$

The **Grassmannian** $\mathbb{G}(k,n)$ is the space parametrizing k-dimensional subspaces of an n-dimensional vector space. For example, $\mathbb{G}(1,n) = \mathbb{P}^{n-1}$.
The Grassmannian $\mathbb{G}(2,4)$ can be embedded into $\mathbb{P}^5$ as a quadric hypersurface. Let V_4 be a four dimensional vector space. The **Plücker embedding** is given by

$$\mathbb{G}(2,V_4) \to \mathbb{P}^5 = \mathbb{P}(\bigwedge^2 V_4), W = \langle w_1, w_2 \rangle \mapsto w_1 \wedge w_2$$

Its image is generated by a Plücker quadric $x_0x_5 - x_1x_4 + x_2x_3$.
Rational parametrization: projection from a point on the quadric.

- Unirational curves/surfaces are rational (Lüroth/Castelnuovo).
- Algebraic varieties which are unirational but NOT rational: Iskovskikh-Manin '71, Clemens-Griffiths '72, Artin-Mumford '72.

K3 surfaces

In the Enriques–Kodaira classification of algebraic surfaces, $K3$ surfaces form one out of four classes with Kodaira dimension 0.

Examples of complete intersections

- Smooth quartic surfaces in $\mathbb{P}^3$, for example, vanishing locus of $x_0^4 + x_1^4 + x_2^4 + x_3^4$, the Fermat quartic.
- Complete intersection of a quadric and a cubic hypersurface in $\mathbb{P}^4$.
- Complete intersecion of three quadric hypersurfaces in $\mathbb{P}^5$.

Definition

A **polarized** $K3$ **surface** is a complete smooth two-dimensional variety S with trivial canonical divisor and vanishing irregularity $h^1(S, \mathcal{O}_S) = 0$ together with a (very) ample line bundle L.
S has a smooth projective model $\mathbb{P}^g = \mathbb{P}(H^0(S, L)^*)$ and has sectional *genus* g and degree $L^2 = 2g - 2$. In particluar, a general hyperplane section is a smooth canonically embedded curve of genus g.

Moduli spaces of K3 surfaces

$\mathcal{F}_g$: the moduli space of polarized $K3$ surfaces of genus g. It is an irreducible quasi-projective variety of dimension 19. It can be identified by a quotient of a hermitian domain of type IV by an arithmetic group.

Example

$\mathcal{F}_3$ is birational to $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)))/\mathrm{PGL}(4)$.
Moduli count: $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)) = 35$ and $\dim \mathrm{PGL}(4) = 15$, hence $35 - 1 - 15 = 19$.
In particular, $\mathcal{F}_3$ is unirational. Similar results for $\mathcal{F}_4$ and $\mathcal{F}_5$.

Theorem (Mukai '88, '89)

The moduli space $\mathcal{F}_g$ is unirational for $g \in \{6, \ldots, 10, 12\}$.

- A general $K3$ surface of genus $g \in \{6, \ldots, 10, 12\}$ is a complete intersection (w.r.t. a vector bundle) on a homogeneous space.
- Explicit unirationality of $\mathcal{F}_g$: the proof of the unirationality provides an explicit algorithm which can be implemented in computer algebra.

Moduli spaces of K3 surfaces of small genus

Example

A general $K3$ surface of genus 8 is a linear section (of codimension 6) of the Grassmannian $\mathbb{G}(2, V_6) \subset \mathbb{P}^{14} = \mathbb{P}(\bigwedge^2 V_6)$. The ideal of $\mathbb{G}(2, V_6)$ is generated by the principal 4×4-Pfaffians of the matrix

$$\begin{pmatrix} 0 & x_0 & x_1 & x_3 & x_6 & x_{10} \\ -x_0 & 0 & x_2 & x_4 & x_7 & x_{11} \\ -x_1 & -x_2 & 0 & x_5 & x_8 & x_{13} \\ -x_3 & -x_4 & -x_5 & 0 & x_9 & x_{13} \\ -x_6 & -x_7 & -x_8 & -x_9 & 0 & x_{14} \\ -x_{10} & -x_{11} & -x_{12} & -x_{13} & -x_{14} & 0 \end{pmatrix},$$

for example, the upper-left Pfaffian is $x_0x_5 - x_1x_4 + x_2x_3$.
The moduli space $\mathcal{F}_8$ of $K3$ surfaces of genus 8 is birational to $\mathbb{G}(6, \bigwedge^2 V_6)/\mathrm{PGL}(6)$.

Moduli spaces of K3 surfaces

Theorem (Gritsenko–Hulek–Sankaran '07)

The Kodaira dimension $\kappa(\mathcal{F}_g)$ is non-negative for $g \in \{41, 43, 44, 47\}$ or $g \geq 49$, and $\mathcal{F}_g$ is of general type for $g \in \{47, 51, 55, 58, 59, 61\}$ or $g \geq 63$.

Theorem (Mukai, Nuer '15, Farkas–Verra '18, '20)

$\mathcal{F}_g$ is unirational for $g \in \{11, 13, 16, 18, 20\}$. $\mathcal{F}_{14}$ and $\mathcal{F}_{22}$ are unirational.

Example (Mukai '96)

Let $\mathcal{P}_{11}$ be the parameter space consisting of pairs $((S, L), C) \in \mathcal{F}_{11} \times \mathcal{M}_{11}$ where $C \in |L|$ is a smooth curve of genus 11. The map $\mathcal{P}_{11} \to \mathcal{M}_{11}$ is birational and hence, unirational.

Explicit construction of K3 surfaces of genus 11

Gushel-Mukai fourfolds and their associated K3 surfaces (work in progress with G. Staglianò)

- *(ordinary) Gushel-Mukai fourfold:*

$$X = Q \cap \underbrace{H \cap \mathbb{G}(2,5)}_{Y \subset \mathbb{P}^8 \text{ delPezzo } 5-\text{fold}} \subset \mathbb{P}^9$$

  a smooth quadratic section of a 5-dimensional linear section of the Grassmannian $\mathbb{G}(2,5) \subset \mathbb{P}^9$.

- *moduli space* $\mathcal{GM}_4$: 24-dimensional, and contains a countable infinity of hypersurfaces

$$\bigcup_d (\mathcal{GM}_4)_d$$

  parametrizing *Hodge-special* GM fourfolds.The lattice spanned by algebraic 2-cycles of rank 3 has *discriminant d* .

Rationality of GM fourfolds

- All GM fourfolds are unirational.
- For some values of the discriminant d, a fourfold $X \in (\mathcal{GM}_4)_d$ has an associated $K3$ surface of degree d.

The notion of associated $K3$ surface leads to the following conjecture.

Conjecture

A GM fourfold $X \in \mathcal{GM}_4$ is rational if and only if it has an associated $K3$ surface.

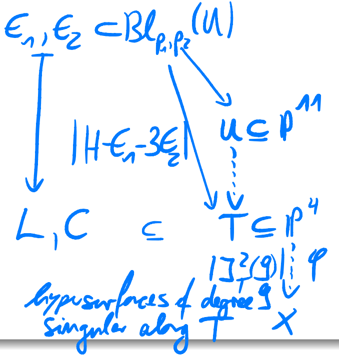
Known rationality: Any GM fourfold in $(\mathcal{GM}_4)_{10}$ is rational (Roth '49).

Theorem [H., Staglianò '20]

Any GM fourfold in $(\mathcal{GM}_4)_{20}$ is rational.

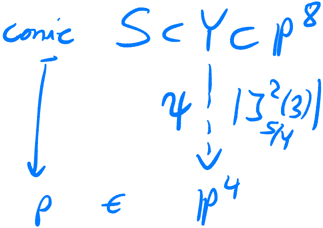
Idea of the rationality proof:

- $U \subset \mathbb{P}^{11}$ a $K3$ surface of genus 11
- π: simple and triple projection of U
  T: singular surface of degree 10 and sectional genus 8
  L: exceptional line
  C: exceptional twisted cubic
- $\varphi : \mathbb{P}^4 \dashrightarrow X$ is a birational map to $X \in (\mathcal{GM}_4)_{20}$.



Inverse of φ

- Y: del Pezzo fivefold
- S: rational surfaces of degree 9 and sectional genus 2
- $\psi : Y \dashrightarrow \mathbb{P}^4$: general fiber is a conic which is a 3-secant to S
- For $S \subset X \subset Y$, the restriction $\psi|_X$ is the inverse of φ.



Explicit construction of K3 surfaces of genus 11

Unirationality of pairs (Russo–Staglianò, H.–Staglianò)

There is an irreducible, unirational, 25-dimensional family $\mathcal{S} \subset \mathrm{Hilb}_Y$ of rational surfaces of degree 9 and sectional genus 2 inside Y. The incidence variety

$$I = \{(S, X) \mid S \subset X\} \subset \mathcal{S} \times \mathbb{P}(H^0(\mathcal{O}_Y(2)))/\mathrm{PGL}(9)$$

is unirational and the second projection to $(\mathcal{GM}_4)_{20}$ is dominant.

Construction of a minimal $K3$ surface of genus 11

- Consider $\psi : Y \dashrightarrow \mathbb{P}^4$ given by $|H^0(\mathcal{I}^2_{S,Y}(3))|$.
- Compute the inverse of $\psi|_X$ whose base locus is a singular surface T of degree 10 and sectional genus 8 (which should be a triple and simple projection of a $K3$ surface of genus 11).
- *Observation: the surface T depends on the pair (S, X), but its exceptional curves L and C only depend on S but not on X.*
- Compute L and C and the rational map of T to $\mathbb{P}^{11}$.

The moduli space $\mathcal{F}_g$

The moduli space $\mathcal{F}_g$ of polarized $K3$ surfaces of genus g behaves similar to the moduli space $\mathcal{M}_g$ of curves of genus g in the following sense. They are (uni-)rational for small values of g and of general type for large values of g. But

Remark to $\mathcal{M}_{15}$

The moduli space of curves of genus 15 is the smallest genus where its unirationality is not known. There exists a unirational subvariety of $\mathcal{M}_{15}$ of codimension 3.

Theorem (Beauville '21, Johnson–Knutsen '04, H.–Mezzedimi)

There are (at least) four unirational hypersurfaces in $\mathcal{F}_g$ for any genus g.