Fifth annual conference
of the SFB-TRR 195

Towards the Classification of Symplectic Linear Quotient Singularities Admitting a Symplectic Resolution

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TU Kaiserslautern
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Joint work with
Gwyn Bellamy and Ulrich Thiel
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Symplectic Groups

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Example

For $V = \mathbb{C}^{2n}$ and $\omega(v, w) := v^\top J_n w$ with $J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, we have

$$\mathrm{Sp}_{2n}(\mathbb{C}) = \{g \in \mathrm{GL}_{2n}(\mathbb{C}) \mid g^\top J_n g = J_n\} \ .$$

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Example

Let $\mathfrak{h}$ be a vector space. Then $\mathfrak{h} \oplus \mathfrak{h}^*$ is symplectic via

$$\omega\big((v, f), (w, g)\big) = g(v) - f(w) \ .$$

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Fact: $\mathrm{Sp}(V) \leq \mathrm{SL}(V)$.

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Let $\mathbf{C}_2 := \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \leq \mathrm{Sp}_2(\mathbb{C})$. Then

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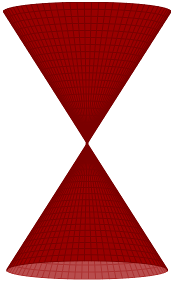

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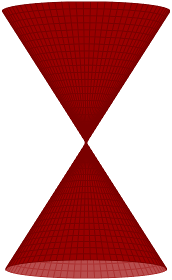

Classical fact

The variety V/G is smooth if and only if G is generated by reflections, i.e. $g \in \mathrm{GL}(V)$ with $\mathrm{rk}(g - 1) = 1$.

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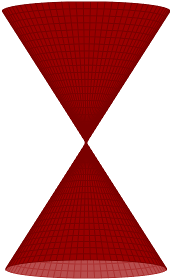

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Corollary

If V is symplectic and $G \leq \mathrm{Sp}(V)$, then V/G is singular.

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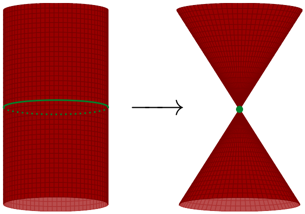

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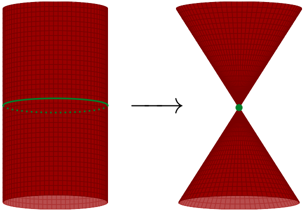

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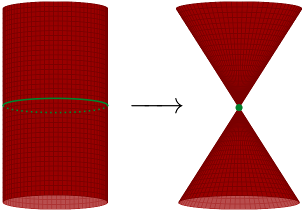

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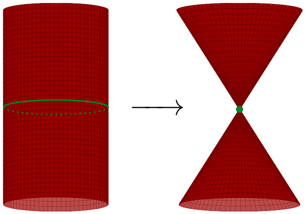

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In general, those do not exist!

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If V/G admits a symplectic resolution, then G is generated by symplectic reflections, i.e. $g \in G$ with $\mathrm{rk}(g - 1) = 2$.

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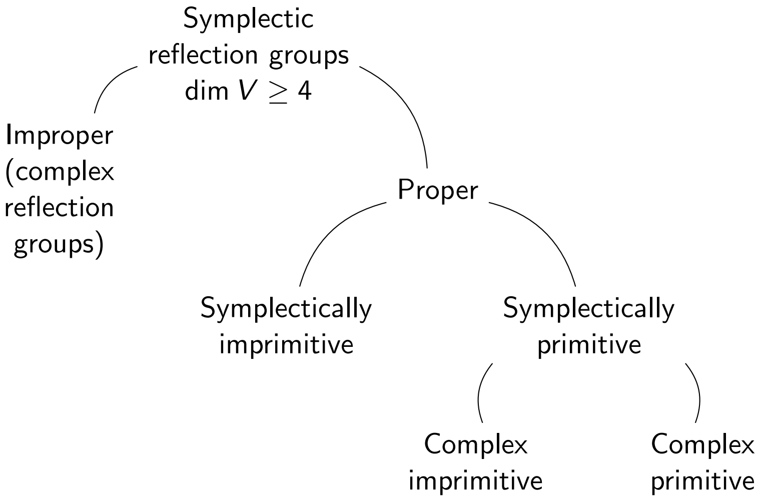
Example

Let $W \leq \mathrm{GL}(\mathfrak{h})$ be a complex reflection group. Then $W \leq \mathrm{Sp}(\mathfrak{h} \oplus \mathfrak{h}^*)$ is a symplectic reflection group.

Symplectic Reflection Groups

Classification by Cohen, 1980



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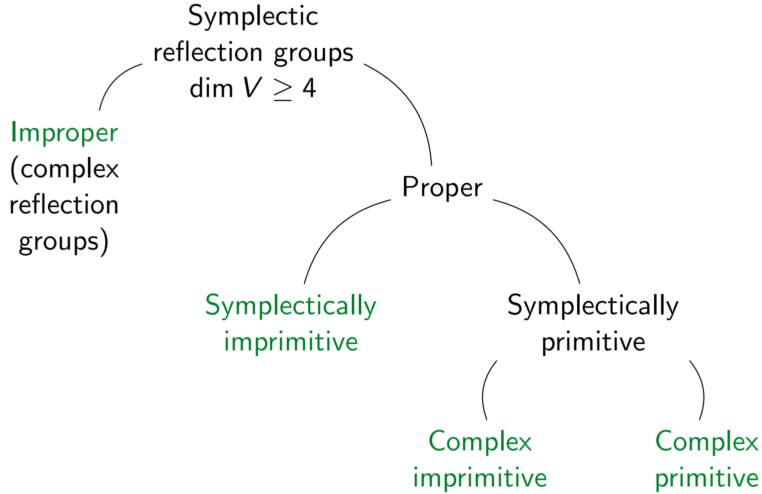

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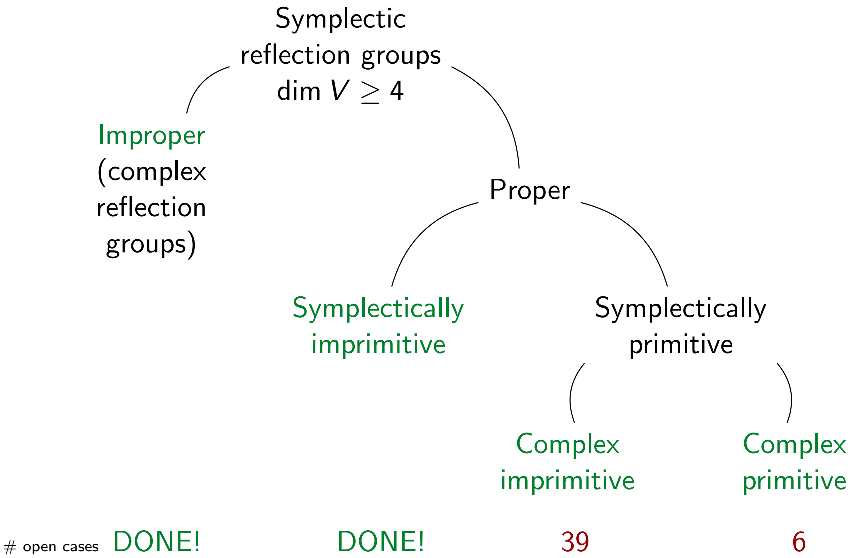

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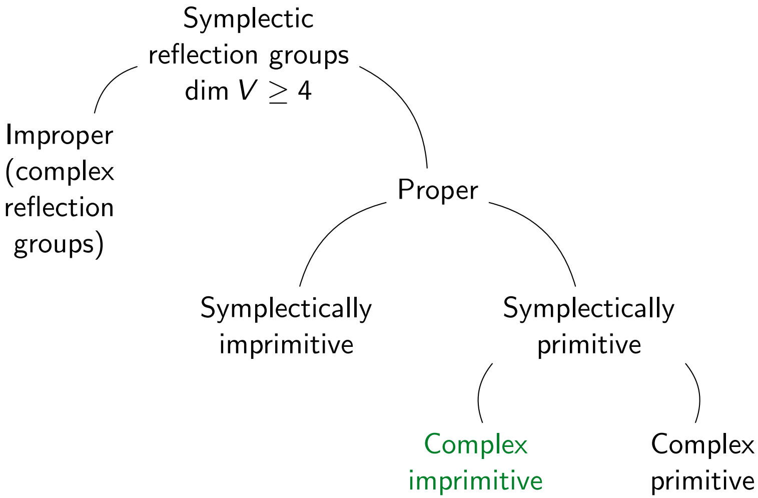

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Symplectically Primitive, Complex Imprimitive Groups

We consider groups G which are

- symplectically primitive, so there is **no** non-trivial decomposition $V = V_1 \oplus \cdots \oplus V_k$ into symplectic subspaces such that for any $g \in G$ and any i there is j with $g(V_i) = V_j$;
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Four infinite families of groups $H \leq \mathrm{GL}_2(\mathbb{C})$, e.g. $\mu_{6d}\mathrm{T}$, $d \in \mathbb{Z}_{\geq 1}$, leading to $E(H) \leq \mathrm{Sp}_4(\mathbb{C})$ generated by

$$h^\vee := \begin{pmatrix} h & 0 \\ 0 & (h^\top)^{-1} \end{pmatrix} \text{ for } h \in H, \text{ and } s := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}.$$

Subgroup Structures

Let $H_0 \leq H$ be the largest complex reflection subgroup.

Lemma

The group H_0 is primitive (e.g. G_5 for $H = \mu_6\mathrm{T}$) and $H_0^\vee$ is a normal subgroup of $E(H)$.

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Any $E(H)$ contains a dihedral group D_d as normal subgroup.

Write $\mathcal{S}(G) \subseteq G$ for the subset of symplectic reflections.

Lemma

We have $\mathcal{S}(E(H)) = \mathcal{S}(H_0^\vee) \dot{\cup} \mathcal{S}(D_d)$, stable under $E(H)$-conjugacy.

Symplectic Reflection Algebras

Deep link to representation theory:
Consider deformations of $\mathbb{C}[V] \rtimes G$, called the symplectic reflection algebras $\mathsf{H}_{\mathbf{c}}(V, G)$.

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Etingof–Ginzburg, 2002; Ginzburg–Kaledin, 2004

If V/G admits a symplectic resolution, then there exists $\mathbf{c}$ such that $\dim S = |G|$ for all irreducible $\mathsf{H}_{\mathbf{c}}(V, G)$-modules S.

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The same strategy was already used for the "improper" symplectic reflection groups.

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That means we can "split up" $\mathbf{c}$ into two parameters

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This gives (sub)algebras

$$\mathsf{H}_{\mathbf{c}_1}(H_0^\vee) \subseteq \mathsf{H}_{\mathbf{c}_1}(H^\vee) \subseteq \mathsf{H}_{\mathbf{c}_1}(E(H)) \leftrightsquigarrow \mathsf{H}_{\mathbf{c}}(E(H)) \leftrightsquigarrow \mathsf{H}_{\mathbf{c}_2}(D_d) \ .$$

Construction of the Module

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$\cap|$

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$\updownarrow$

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Remember: We want to construct an $\mathsf{H}_{\mathbf{c}}(E(H))$-module of dimension $\neq |E(H)|$.

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This is an $\mathsf{H}_{\mathbf{c}}(E(H))$-module if and only if all the constituents of $M|_{D_d}$ are $\mathbf{c}_2$-rigid, i.e. if they are isomorphic to a simple $\mathsf{H}_{\mathbf{c}_2}(D_d)$-module.

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For $\lambda \in \mathrm{Irr}(H)$, we have $\lambda|_{H_0} \in \mathrm{Irr}(H_0)$ giving rise to a simple $\mathsf{H}_{\mathbf{c}_1}(H_0^\vee)$-module $L(\lambda|_{H_0})$ with $\dim L(\lambda|_{H_0}) \leq |H_0|$.

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Step 2: Reduction to H_0.

For $\lambda \in \mathrm{Irr}(H)$, we have $\lambda|_{H_0} \in \mathrm{Irr}(H_0)$ giving rise to a simple $\mathsf{H}_{\mathbf{c}_1}(H_0^\vee)$-module $L(\lambda|_{H_0})$ with $\dim L(\lambda|_{H_0}) \leq |H_0|$.

Lemma

The module $L(\lambda|_{H_0})$ is a simple $\mathsf{H}_{\mathbf{c}_1}(H^\vee)$-module as well.

$$\begin{array}{c} \mathsf{H}_{\mathbf{c}_1}(H_0^\vee) \\ \rotatebox{90}{$\subseteq$} \\ \mathsf{H}_{\mathbf{c}_1}(H^\vee) \\ \rotatebox{90}{$\subseteq$} \\ \mathsf{H}_{\mathbf{c}_1}(E(H)) \\ \updownarrow \\ \mathsf{H}_{\mathbf{c}}(E(H)) \\ \updownarrow \\ \mathsf{H}_{\mathbf{c}_2}(D_d) \end{array}$$

Results

Conclusion

We can construct an $\mathsf{H}_{\mathbf{c}}(E(H))$-module M with $\dim M < |E(H)|$, if we can find λ such that $L(\lambda|_{H_0})|_{D_d}$ is $\mathbf{c}_2$-rigid and $H_0 \lneq H$.

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Such a module exists except in possibly 73 cases.

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Such a module exists except in possibly 39 cases. In at least 18 of them no such module exists.

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Final result (so far)

Let $G \leq \mathrm{Sp}(V)$ be a symplectically primitive complex imprimitive symplectic reflection group. Then the corresponding quotient V/G does not admit a symplectic resolution except in possibly 39 cases.