Orders and Polytropes: Matrix Algebras from Valuations

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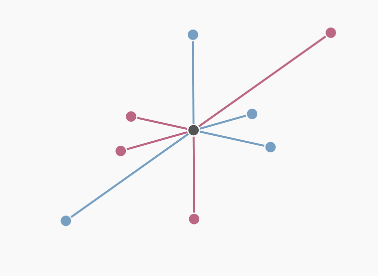

TRR 195
SYMBOLIC TOOLS

Max-Planck-Institut für
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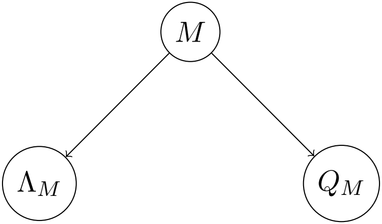
Intro 〉 The general idea

Let $d \in \mathbb{Z}_{>0}$ and let $M = (m_{ij}) \in \mathbb{Z}^{d\times d} = \mathrm{Mat}_d(\mathbb{Z})$.



$\mathcal{O}_K$-module in $K^{d\times d}$ polytope in $\mathbb{R}^{d-1}$

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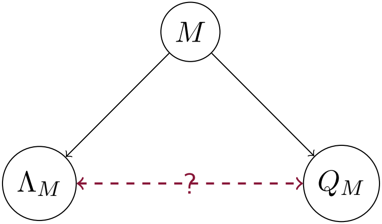

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Goal: understand what the interplay between these objects is.

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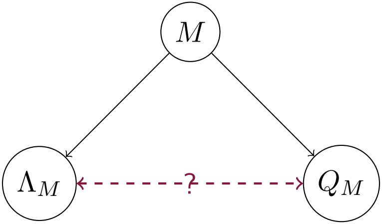

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Goal: understand what the interplay between these objects is.

This connection is most strong when Λ_M is an order.

Let K be a field with a surjective valuation map

$$\mathrm{val} : K \to \mathbb{Z} \cup \{\infty\}.$$

Example. $K = \mathbb{Q}$ and $\mathrm{val} = \mathrm{val}_3$ is the 3-adic valuation. Then

$$\mathrm{val}(45) = \mathrm{val}(3^2 5) = 2, \quad \mathrm{val}(13/27) = \mathrm{val}(3^{-3} 13) = -3.$$

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Denote

- $\mathcal{O}_K = \{x \in K : \mathrm{val}(x) \geq 0\}$ is the valuation ring of K,
- $\mathfrak{m}_K = \{x \in K : \mathrm{val}(x) > 0\} \triangleleft \mathcal{O}_K$ unique maximal,
- $\pi \in K$ such that $\mathrm{val}(\pi) = 1$ is a uniformizer.

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Then $\mathfrak{m}_K = (\pi)$ and every ideal of $\mathcal{O}_K$ is of the form $\mathfrak{m}_K^k = (\pi^k)$.
If $\mathrm{val} = \mathrm{val}_p$ is the p-adic valuation, then $\pi = p$.

The valuation val can be extended to K^d or $K^{d\times d}$ coordinate-wise:

$$\mathrm{val}_3(2, 15, -1/12) = (0, 1, -1), \quad \mathrm{val}_3 \begin{pmatrix} 0 & 66 \\ -1/3 & 7 \end{pmatrix} = \begin{pmatrix} \infty & 1 \\ -1 & 0 \end{pmatrix}$$

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Then the set

$$\Lambda_M = \{X \in K^{d\times d} : \mathrm{val}(X) \geq M\}$$

is an $\mathcal{O}_K$-module because in K:

- $\mathrm{val}(x + y) \geq \min\{\mathrm{val}(x), \mathrm{val}(y)\}$
- $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$

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- $\mathrm{val}(\alpha y_{ij}) = \mathrm{val}(\alpha) + \mathrm{val}(y_{ij}) \geq 0 + m_{ij} = m_{ij}$

Remark. Λ_M has maximal rank d^2 as a free $\mathcal{O}_K$-submodule of $K^{d\times d}$ and it lives in a ring!

Example. For $K = \mathbb{Q}$, $d = 3$, and $\mathrm{val} = \mathrm{val}_p$:

$$\mathrm{val}_p \underbrace{\begin{pmatrix} 1 & 1 & p \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{M} \text{ but } \begin{pmatrix} \star & \star & 0 \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} = \mathrm{val} \underbrace{\begin{pmatrix} 2+p & 2+p & 1+2p \\ 3 & 3 & 2+p \\ 3 & 3 & 2+p \end{pmatrix}}_{X^2}$$

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Example. $M = \mathbf{0}_{d\times d} \Rightarrow \Lambda_M = \mathcal{O}_K^{d\times d}$ is a maximal order in $K^{d\times d}$!

Definition.

- A $(\mathcal{O}_K$-)lattice in K^d is a free $\mathcal{O}_K$-submodule of rank d.
- An order in $K^{d\times d}$ is a lattice in $K^{d\times d}$ that is also a ring.

Remark. Orders of the form Λ_M are called graduated, tiled, split or monomial by different authors.

Proposition (Plesken). Λ_M is an order if and only if

$$m_{ii} = 0,\ m_{ij} + m_{jk} \geq m_{ik} \text{ for } 1 \leq i, j, k \leq d. \qquad (1)$$

Remark. If $m_{ii} = 0$, we write $M \in \mathbb{Z}_0^{d\times d}$. From now on we will often implicitly assume that (1) holds.

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Example.

- The $\mathcal{O}_K^{d\times d}$-stable lattices are $\{\pi^n L_0 : n \in \mathbb{Z}\}$. (maximality)
- If L, L' are Λ_M-stable, then $L \cap L'$ and $L + L'$ are Λ_M-stable.

Proposition (Plesken). A lattice L in K^d is stable under Λ_M if and only if there exists $u \in \mathbb{Z}^d$ with

$$u_i - u_j \leq m_{ij} \quad \text{for} \quad 1 \leq i, j \leq d, \tag{2}$$

such that $L = L_u = \mathcal{O}_K \pi^{u_1} e_1 \oplus \ldots \oplus \mathcal{O}_K \pi^{u_d} e_d$. Moreover, two stable lattices L_u and $L_{u'}$ are isomorphic as Λ_M-modules if and only if there exists $n \in \mathbb{Z}$ such that $L_{u'} = \pi^n L_u$.

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Remark. Let $u \in \mathbb{Z}^d$ and set $M(u) = (u_i - u_j)$.

- Lattices with $L' = \pi^n L$ are called equivalent, denoted $L \sim L'$
- Lattices of the form L_u are called diagonal and all have compatible bases
- $\mathrm{End}_{\mathcal{O}_K}(L_u) = \Lambda_{M(u)} = \{X \in K^{d \times d} : \mathrm{val}(X) \geq M(u)\}$

Let $\Gamma = \{L_1, \ldots, L_n\}$ be a finite set of lattices in K^d. Then the Plesken-Zassenhaus ring

$$\mathrm{PZ}(\Gamma) \ = \ \mathrm{End}_{\mathcal{O}_K}(L_1) \cap \cdots \cap \mathrm{End}_{\mathcal{O}_K}(L_n)$$

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Remark.

- Each L_i is $\mathrm{PZ}(\Gamma)$-stable.
- Any $L_i \cap L_j$ and $L_i + L_j$ is $\mathrm{PZ}(\Gamma)$-stable.
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Proposition (Plesken). If Γ consists of diagonal lattices, then $\mathrm{PZ}(\Gamma)$ is graduated. Moreover, the converse is also true.

The min-plus and max-plus algebras $(\mathbb{R}, \underline{\oplus}, \odot)$ and $(\mathbb{R}, \overline{\oplus}, \odot)$ are defined by the operations

$$a \,\underline{\oplus}\, b = \min\{a, b\}, \quad a \,\overline{\oplus}\, b = \max\{a, b\}, \quad a \odot b = a + b.$$

Example. $L_u \cap L_{u'} = L_{u \,\overline{\oplus}\, u'}$ and $L_u + L_{u'} = L_{u \,\underline{\oplus}\, u'}$

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These operations induce also product of matrices.

Example. If $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $N = \begin{pmatrix} -2 & 0 \\ 3 & 1 \end{pmatrix}$, and $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then

$$M \,\overline{\odot}\, N = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}, \qquad M \,\overline{\odot}\, u = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$
$$M \,\underline{\odot}\, N = \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix}, \qquad M \,\underline{\odot}\, u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

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Proposition.

- Λ_M is an order if and only if $M \odot M = M$
- L_u is a stable lattice if and only if $M \odot u^{\mathrm{t}} \geq u^{\mathrm{t}}$
- L_u is Λ_M-stable iff $[u] \in Q_M \cap (\mathbb{Z}^d/\mathbb{Z}\mathbf{1})$ and actually

$$\{[L] : L \text{ is } \Lambda_M\text{-stable}\} \longleftrightarrow Q_M \cap (\mathbb{Z}^d/\mathbb{Z}\mathbf{1})$$

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Theorem. Let $\Gamma = \{L_{u^{(1)}}, \ldots, L_{u^{(n)}}\}$ be any configuration of diagonal lattices in K^d. Then $\mathrm{PZ}(\Gamma) = \Lambda_M$ where

$$M = M(u^{(1)}) \mathbin{\overline{\oplus}} M(u^{(2)}) \mathbin{\overline{\oplus}} \cdots \mathbin{\overline{\oplus}} M(u^{(n)}).$$

Definition.

- $\mathcal{P}_d = \{N \in \mathbb{R}_0^{d\times d} : N \underline{\odot} N = N\}$ is the polytrope region
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Remark.

- $\mathcal{P}_d$ is a $(d^2 - d)$-dimensional convex polyhedral cone, defined by "idempotence" inequalities $m_{ik} \leq m_{ij} + m_{jk}$
- The number of facets of $\mathcal{P}_d$ is $d(d-1)(d-2)$
- For $d = 3, 4, 5$ we compute f-vectors and vertices of $\mathcal{P}_d$
- $\mathcal{P}_d(M)$ parametrizes all subpolytropes of Q_M
- $\mathcal{P}_d(M)$ parametrizes all $\mathcal{O}_K$-orders containing Λ_M

Indeed: $N \leq M \iff Q_N \subseteq Q_M \iff \Lambda_N \supseteq \Lambda_M$.

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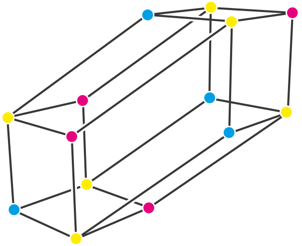
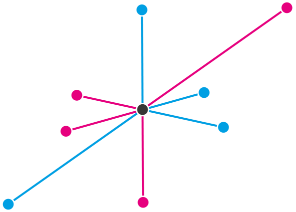
Theorem. Let $M \in \mathcal{P}_d$ be in standard form. Then Q_M is both a min-plus and a max-plus simplex. The min-plus vertices u are the columns of M and represent L_u's that are projective Λ_M-modules. The max-plus vertices v are the columns of $-M^t$, and they represent the injective Λ_M-modules L_v.





Here $d = 4$ and $M = J_4$ has all 1's outside the diagonal.

For M in standard form, the nonzero fractional ideals of Λ_M are

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Remark.

- $N \in \mathbb{Z}_0^{d\times d} \cap \mathcal{Q}_M \Rightarrow I_N = \Lambda_N$ is an order
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Definition. The ideal class group $\mathcal{G}_M$ of M is the maximal subgroup of the semigroup $\mathcal{Q}_M$.

Example. $\mathcal{G}_{J_2} \cong \mathbb{Z}/2\mathbb{Z},\; \mathcal{G}_{J_3} \cong \mathbb{Z}/6\mathbb{Z},\; \mathcal{G}_{J_4} \cong \mathrm{S}_4$

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Question. How does this sequence continue?

- Graduated orders can be realized as Plesken-Zassenhaus rings of configurations of diagonal lattices and viceversa.
- Not every order in $K^{d\times d}$ is graduated (or conjugate to a graduated order). It is however true that every graduated order is isomorphic to one in standard form.
- Even more generally, not every order in $K^{d\times d}$ is an intersection of maximal orders.

For more on graduated orders: Plesken, *Group rings of finite groups over p-adic integers* (1983).

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Question. If we allow non-diagonal lattices, what are the orders describing the Plesken-Zassenhaus ring and their associated stable lattices?

Definition. The Bruhat-Tits building $\mathcal{B}_d(K)$ is a simplicial complex where:

- the vertices are equivalence classes of lattices in K^d,
- $([L_1], \dots, [L_s])$ is a simplex if $L_1 \supset L_2 \supset \dots L_s \supset pL_1$ (up to reordering and picking representatives)

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- the vertices are equivalence classes of lattices in K^d,
- $([L_1], \ldots, [L_s])$ is a simplex if $L_1 \supset L_2 \supset \ldots L_s \supset pL_1$ (up to reordering and picking representatives)

Remark. From the point of view of the building, looking at diagonal lattices is the same as working in one apartment (compatible bases)

Remark. If $\mathrm{PZ}(\Gamma) = \Lambda_M$, then Q_M describes the geodesic convex hull of Γ in the building.

Question. What is life like when you are not quarantined in one apartment?

And don't forget to check out our Mathrepo page:
https://mathrepo.mis.mpg.de/OrdersPolytropes/index.html