

conley

COMPUTING CONNECTION MATRICES IN Maple

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ABSTRACT. In this work we announce the Maple package `conley` to compute connection and C -connection matrices. `conley` is based on our abstract homological algebra package `homalg`. We emphasize that the notion of braids is irrelevant for the definition and for the computation of such matrices. We introduce the notion of triangles that suffices to state the definition of (C) -connection matrices. The notion of octahedra, which is equivalent to that of braids is also introduced.

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1. INTRODUCTION

The algebraic theory of connection and C -connection matrices is concerned with condensing the information of a graded module octahedron in a matrix called a connection matrix or C -connection matrix.

In Section 2 we recall the central notions of this algebraic theory, introduce some new notions and give a *braid-free* definition of (C)-connection matrices. For this we introduce the notion of a graded module triangular system or simply a graded module triangle. This is the only notion used in the definition of connection and C -connection matrices.

The role of braids is emphasized in formulating the necessary and sufficient conditions to the existence of C -connection matrices in Section 3. There we also introduce the equivalent notion of octahedra.

The notion of symmetric C -connection matrix is defined in Section 4.

In Section 5 we briefly review the definition of the homology Conley index of an isolated invariant set and the theory of Morse decompositions of a dynamical system. This theory is the main source of applications of connection matrices.

The package `conley` is more or less a tiny application of our more elaborate homological algebra package `homalg` [BR]. In Section 6 some technical comments on `conley` are made.

Finally, in Section 7 we present some simple examples using `conley`.

We are very much indebted to STANISLAUS MAIER-PAAPE, who initiated the joint Conley index seminar in Aachen, introduced us to the subject and explained us the fascinating dynamical side of the theory. We would also like to thank him for fruitful discussions which revealed a serious mistake in the first version of this paper. We also would like to thank all the participants of the seminar, and especially MAX NEUNHOEFFER and FELIX NOESKE for discussions about the algebraic aspects of the theory.

Aspects of dynamical systems are not in the focus of this paper. However, joint work with MAIER-PAAPE on more substantial dynamical applications and in particular on transition matrices [MM02, MPMW] is in preparation. Transition matrices can also be computed using `conley`. We don't touch upon this theory here.

Note that we apply morphisms from the *right* and hence we use the *row convention* for matrices. As one consequence we talk about *lower* triangular instead of upper triangular matrices.

2. CONNECTION MATRICES

2.1. Posets. A set P together with a strict partial order $>$ (i.e. an irreflexive and transitive relation $> \subset P \times P$) is called a *poset* and is denoted by $(P, >)$.

A subset $I \subset P$ is called an *interval* in $(P, >)$ if for all $p, q \in I$ and $r \in P$ the following implication holds:

$$q > r > p \quad \Rightarrow \quad r \in I.$$

The set of all intervals in $(P, >)$ is denoted by $\mathcal{J}(P, >)$.

An n -tuple (I_1, \dots, I_n) of intervals in $(P, >)$ is called *adjacent* if these intervals are mutually disjoint, $\bigcup_{i=1}^n I_i$ is an interval in $(P, >)$ and for all $p \in I_j$, $q \in I_k$ the following

implication holds:

$$j < k \quad \Rightarrow \quad p \not\prec q.$$

The set of all adjacent n -tuples of intervals in $(P, >)$ is denoted by $\mathcal{J}_n(P, >)$.

If (I_1, \dots, I_n) is an adjacent n -tuple of intervals in $(P, >)$, then set $I_1 I_2 \dots I_n := \bigcup_{i=1}^n I_i$.

If $(I, J) \in \mathcal{J}_2(P, >)$ as well as $(J, I) \in \mathcal{J}_2(P, >)$, then I and J are said to be *noncomparable*.

2.2. Triangles. For notions like quasi-isomorphism and distinguished triangles we refer to [GM03].

Definition 2.1 (Graded module triangle). Let $(P, >)$ be a poset. A *graded module triangular system* or simply a *graded module triangle* G over $(P, >)$ consists of graded modules $G(I)$, $I \in \mathcal{J}(P, >)$, and homomorphisms of graded modules

$$\begin{aligned} i(I, IJ) : G(I) &\longrightarrow G(IJ), \\ p(IJ, J) : G(IJ) &\longrightarrow G(J), \\ \partial(J, I) : G(J) &\longrightarrow G(I) \end{aligned}$$

for all $(I, J) \in \mathcal{J}_2(P, >)$, where $i(I, IJ)$ and $p(IJ, J)$ are of degree 0, $\partial(J, I)$ is of degree -1 , and the following two conditions are satisfied:

- a) For all $(I, J) \in \mathcal{J}_2(P, >)$ the triangle

$$\begin{array}{ccc} & G(IJ) & \\ i(I, IJ) \nearrow & & \searrow p(IJ, J) \\ G(I) & \xleftarrow{\partial(J, I)} & G(J) \end{array}$$

is distinguished.

- b) If $I, J \in \mathcal{J}(P, >)$ are noncomparable, then $i(I, IJ)p(JI, I) = \text{id}_{G(I)}$.

We think of a graded module as an element of $\text{Kom}_0(\mathcal{A}) \subset \text{Kom}(\mathcal{A})$. $\text{Kom}(\mathcal{A})$ is the category of complexes over an abelian category \mathcal{A} and $\text{Kom}_0(\mathcal{A})$ the complete subcategory of cyclic complexes, i.e. complexes with zero boundary maps. Cf. [GM03, III.2.3].

The one dimensional unraveling of condition a) is the following long exact sequence condition:

- a') For all $(I, J) \in \mathcal{J}_2(P, >)$

$$\dots \rightarrow G(I) \xrightarrow{i(I, IJ)} G(IJ) \xrightarrow{p(IJ, J)} G(J) \xrightarrow{\partial(J, I)} G(I) \rightarrow \dots$$

is a long exact sequence.

Definition 2.2 (Homomorphism of graded module triangles). Let $(P, >)$ be a poset and let G and G' be graded module triangles over $(P, >)$.

- a) A *homomorphism of graded module triangles* $\theta : G \rightarrow G'$ consists of homomorphisms of graded modules $\theta(I) : G(I) \rightarrow G'(I)$, $I \in \mathcal{J}(P, >)$, such that the following diagram

commutes for all $(I, J) \in \mathcal{J}_2(P, >)$:

$$(1) \quad \begin{array}{ccccc} & & G(IJ) & & \\ & i(I, IJ) \nearrow & \downarrow \theta(IJ) & \searrow p(IJ, J) & \\ G(I) & \longleftarrow & & \longrightarrow & G(J) \\ & & \downarrow \partial(J, I) & & \\ & & G'(IJ) & & \\ \theta(I) \downarrow & i'(I, IJ) \nearrow & & \searrow p'(IJ, J) & \downarrow \theta(J) \\ G'(I) & \longleftarrow & & \longrightarrow & G'(J) \\ & & \downarrow \partial'(J, I) & & \end{array}$$

- b) An *isomorphism of graded module triangles* $\theta : G \rightarrow G'$ is a homomorphism of graded module triangles where all $\theta(I)$ are isomorphisms of graded modules.

The two dimensional unraveling of the above diagram is:

$$(2) \quad \begin{array}{cccccccc} \cdots & \longrightarrow & G(I) & \xrightarrow{i} & G(IJ) & \xrightarrow{p} & G(J) & \xrightarrow{\partial} & G(I) & \longrightarrow & \cdots \\ & & \downarrow \theta(I) & & \downarrow \theta(IJ) & & \downarrow \theta(J) & & \downarrow \theta(I) & & \\ \cdots & \longrightarrow & G'(I) & \xrightarrow{i'} & G'(IJ) & \xrightarrow{p'} & G'(J) & \xrightarrow{\partial'} & G'(I) & \longrightarrow & \cdots \end{array}$$

Definition 2.3 (Chain complex triangle). Let $(P, >)$ be a poset. A *chain complex triangular system* or simply a *chain complex triangle* C over $(P, >)$ consists of chain complexes $C(I)$, $I \in \mathcal{J}(P, >)$, and chain maps $i(I, IJ) : C(I) \rightarrow C(IJ)$ and $p(IJ, J) : C(IJ) \rightarrow C(J)$ for all $(I, J) \in \mathcal{J}_2(P, >)$, satisfying the following two conditions:

- The complex $0 \rightarrow C(I) \xrightarrow{i(I, IJ)} C(IJ) \xrightarrow{p(IJ, J)} C(J) \rightarrow 0$ is quasi-isomorphic¹ to a distinguished triangle of complexes.
- If $I, J \in \mathcal{J}(P, >)$ are noncomparable, then $i(I, IJ)p(JI, I) = \text{id}_{C(I)}$.

Short exact and short weakly exact sequences are examples of complexes quasi-isomorphic to a distinguished triangle of complexes. A sequence $0 \rightarrow C' \xrightarrow{i} C \xrightarrow{p} C'' \rightarrow 0$ of complexes is called short exact if i is injective, $ip = 0$, and $C/\text{im}(i) \rightarrow C''$ induces isomorphism on homology. Using the cylinder-cone-translation construction [GM03, III.3] one can construct distinguished triangles out of such sequences.

Remark 2.4 (Chain complex generated triangles). Let $(P, >)$ be a poset and C a chain complex triangle over $(P, >)$. The homology modules $H(I)$, $I \in \mathcal{J}(P, >)$, together with the

¹More precisely, isomorphic in the localized category, in which one inverts quasi-isomorphisms, where quasi-isomorphisms are chain maps inducing isomorphism on homology [GM03, III.2].

homomorphisms $\iota(I, IJ) : H(I) \rightarrow H(IJ)$ and $\pi(IJ, J) : H(IJ) \rightarrow H(J)$ induced by the chain maps $i(I, IJ)$ and $p(IJ, J)$ form a graded module triangle HC over $(P, >)$.

2.3. C -Connection and connection matrices. We fix a poset $(P, >)$. In what follows, we consider collections $C = \{C(p) \mid p \in P\}$ of graded modules, which are indexed by P , and a homomorphism $\Delta : \bigoplus_{p \in P} C(p) \rightarrow \bigoplus_{p \in P} C(p)$.

For an interval I in $(P, >)$ set $C(I) := \bigoplus_{p \in I} C(p)$ and denote by $\Delta(I)$ the homomorphism $\iota_I \Delta \pi_I$, where $\iota_I : C(I) \rightarrow C(P)$ is the canonical injection and $\pi_I : C(P) \rightarrow C(I)$ is the canonical projection.

If $p_1, p_2 \in P$, we refer to the restriction of Δ to $C(p_1)$ by $\Delta_{p_1} : C(p_1) \rightarrow C(P)$, and the composition $\Delta_{p_1} \pi_{p_2}$, where π_{p_2} is the projection $C(P) \rightarrow C(p_2)$, is denoted by $\Delta_{p_1, p_2} : C(p_1) \rightarrow C(p_2)$. Then Δ can be visualized as a matrix with Δ_{p_1} as its p_1 -th row and Δ_{p_1, p_2} as its entry at position (p_1, p_2) . The same applies to $\Delta(I)$ for $I \in \mathcal{J}(P, >)$, if $p_1, p_2 \in I$.

Definition 2.5 ([Fra88, Def. 1.3]). Δ being as above:

- Δ is said to be *lower triangular* if $\Delta_{p_1, p_2} \neq 0$ implies $p_1 > p_2$ or $p_1 = p_2$.
- Δ is said to be *strictly lower triangular* if $\Delta_{p_1, p_2} \neq 0$ implies $p_1 > p_2$.
- Δ is called a *boundary map* if each Δ_{p_1, p_2} is a homomorphism of graded modules of degree -1 and $\Delta \circ \Delta = 0$.

Proposition 2.6 ([Fra89, Prop. 3.3]). *Let $C = \{C(p) \mid p \in P\}$ be a collection of graded modules indexed by P and let $\Delta : \bigoplus_{p \in P} C(p) \rightarrow \bigoplus_{p \in P} C(p)$ be a lower triangular boundary map. Then:*

- $C(I)$ and $\Delta(I)$ form a chain complex $C^\Delta(I)$ for all $I \in \mathcal{J}(P, >)$.
- For all $(I, J) \in \mathcal{J}_2(P, >)$, the obvious injection and projection maps $i(I, IJ)$ and $p(IJ, J)$ are chain maps and

$$0 \rightarrow C^\Delta(I) \xrightarrow{i(I, IJ)} C^\Delta(IJ) \xrightarrow{p(IJ, J)} C^\Delta(J) \rightarrow 0$$

is a short exact sequence.

Using the above introduced notion the previous Proposition simply asserts that C^Δ is a chain complex triangle.

The following definition of a connection matrix avoids braids (cf. [Fra89, Def. 3.6]).

Definition 2.7 ((C) -Connection matrix). Let G be a graded module triangle over $(P, >)$, $C = \{C(p) \mid p \in P\}$ a collection of graded modules indexed by P , and $\Delta : \bigoplus_{p \in P} C(p) \rightarrow \bigoplus_{p \in P} C(p)$ a lower triangular boundary map. Let C^Δ be the chain complex triangle formed by the $C(I)$, $\Delta(I)$, $i(I, IJ)$, and $p(IJ, J)$.

- If HC^Δ and G are isomorphic as graded module triangles, then Δ is called a *C -connection matrix* for G .
- If furthermore $C(p) \cong G(p)$, then Δ is called a *connection matrix* for G .

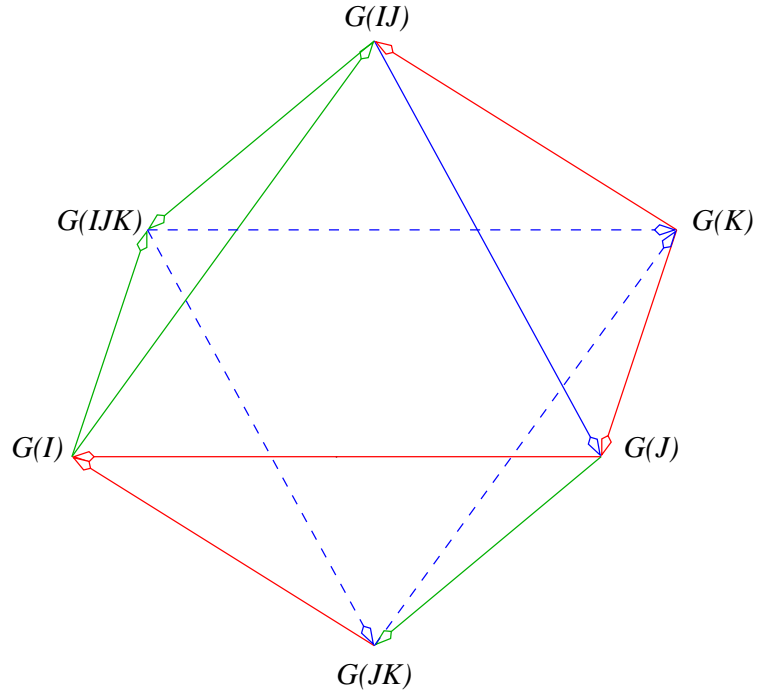
3. OCTAHEDRA AND BRAIDS AND THE EXISTENCE OF C -CONNECTION MATRICES

In this section we want to review the necessary and sufficient conditions a graded module triangle G must satisfy so that a connection matrix (resp. C -connection matrix) for G exists.

3.1. Graded module octahedra.

Definition 3.1 (Graded module octahedron). Let $(P, >)$ be a poset. A *graded module octahedral system* or simply a *graded module octahedron* G over $(P, >)$ is a graded module triangle over $(P, >)$ satisfying the extra condition:

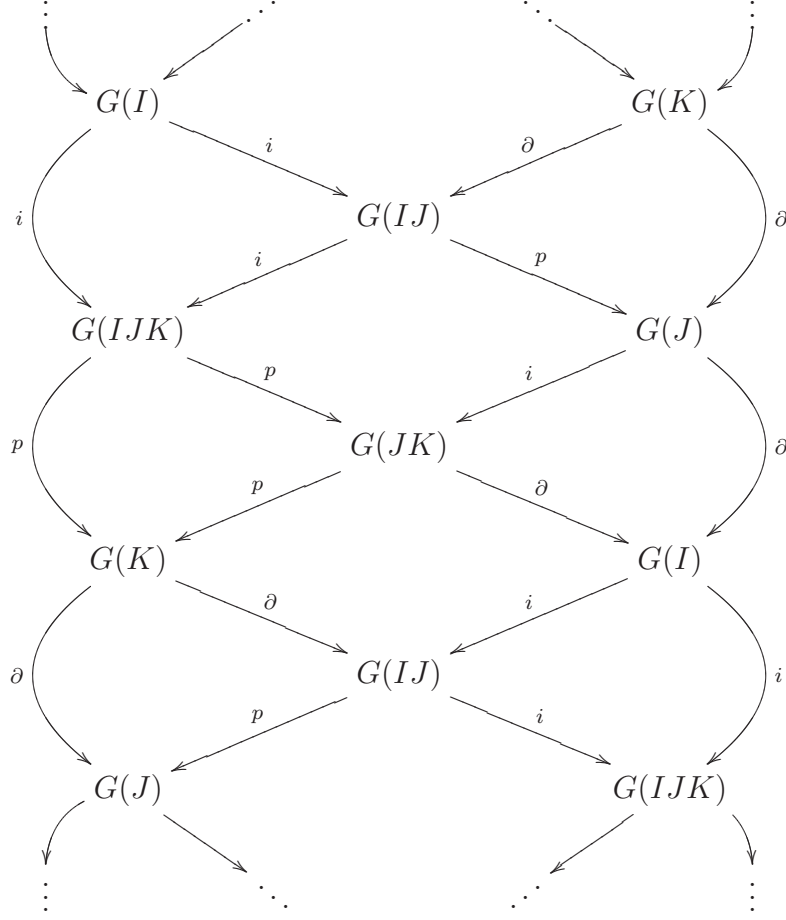
- c) For all $(I, J, K) \in \mathcal{I}_3(P, >)$ the following octahedron commutes and is distinguished:



By “commutes and is distinguished” the following is meant: First notice that there are two types of triangles resp. squares appearing in the octahedron: cyclic and noncyclic ones. For the cyclic ones (4 triangles and one square) we require distinguishedness and for the others (4 triangles and two squares) commutativity. Actually the distinguishedness of the square, which is the horizontal one, follows from the rest.

The two dimensional unraveling of the octahedron condition c) is the following braid condition [Fra88, Def. 1.1]:

c') For all $(I, J, K) \in \mathcal{J}_3(P, >)$ the following braid diagram commutes:



Remark 3.2. It is worth noting that the octahedron appearing in the octahedral property c) of Definition 3.1 (which is equivalent to the braid property c') found by [Fra86] in the context of homology indices of index filtrations of Morse decompositions and exploited by him in [Fra88, Fra89]) is the same as the “octahedron diagram” which is part of the definition of a triangulated category [GM03, IV.1, Axiom TR4].

Definition 3.3 (Homomorphism of graded module octahedra). Let $(P, >)$ be a poset and let G and G' be graded module octahedra over $(P, >)$.

- A *homomorphism of graded module octahedra* $\theta : G \rightarrow G'$ is a homomorphism of the underlying graded module triangles.
- An *isomorphism of graded module octahedra* $\theta : G \rightarrow G'$ is a homomorphism of graded module octahedra where all $\theta(I)$ are isomorphisms of graded modules.

This is equivalent to the definition of homomorphisms and isomorphisms of graded module braids appearing in [Fra88, Def. 1.2].

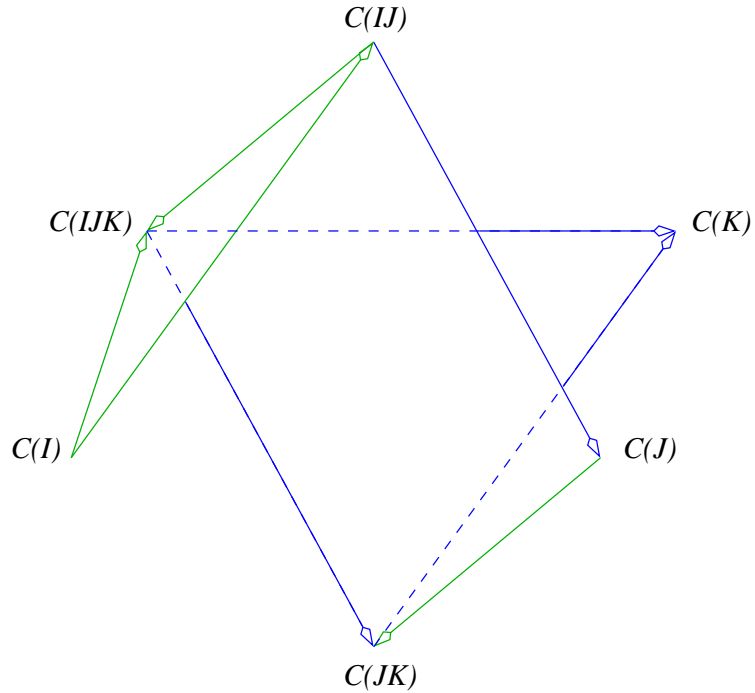
Remark 3.4. One notices that isomorphism of two graded module octahedra (resp. braids) amounts to the isomorphism of the underlying graded module triangles. Note that for

each interval I there is exactly one isomorphism $\theta(I) : G(I) \rightarrow G'(I)$ entering in all the commutative diagrams in (1) (or (2)). This condition will be essential in Subsection 3.3.

3.2. Chain complex octahedra.

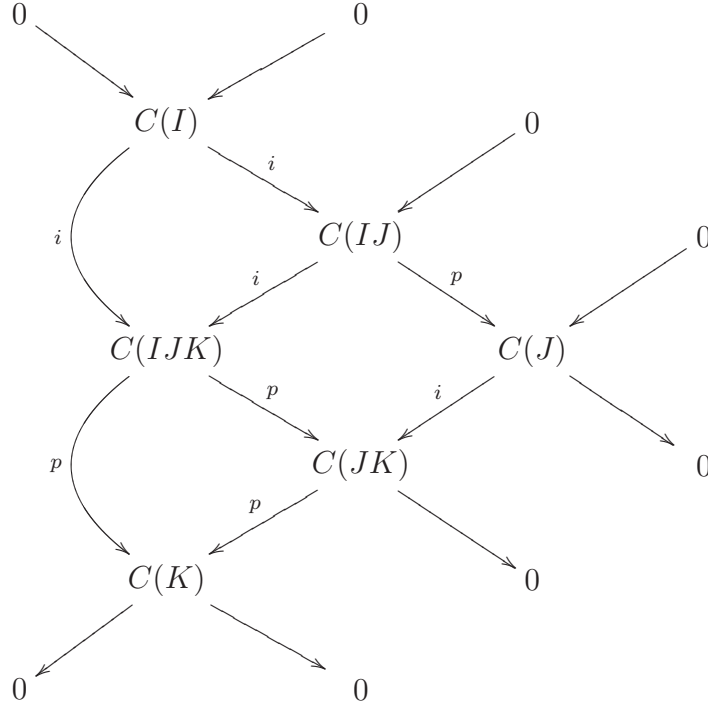
Definition 3.5 (Chain complex octahedron, [Fra89, Def. 2.6]). Let $(P, >)$ be a poset. A *chain complex octahedral system* or simply a *chain complex octahedron* C over $(P, >)$ is a chain complex triangle satisfying the extra condition:

- c) For all $(I, J, K) \in \mathcal{J}_3(P, >)$ the following octahedron commutes and is distinguished:



Since, up to quasi-isomorphism, we are talking about distinguished triangles we can complete the missing edges of the above octahedron and obtain a full octahedron as in Definition 3.1, but now on the level of complexes (up to quasi-isomorphism). This explains again “commutes and is distinguished”.

The two dimensional unraveling of the above octahedron is called a chain complex braid:



Proposition 3.6 ([Fra89, Prop. 2.7]). *Let $(P, >)$ be a poset and C a chain complex octahedron (resp. braid) over $(P, >)$. The homology modules $H(I)$, $I \in \mathcal{J}(P, >)$, together with the homomorphisms $\iota(I, IJ) : H(I) \rightarrow H(IJ)$ and $\pi(IJ, J) : H(IJ) \rightarrow H(J)$ induced by the chain maps $i(I, IJ)$ and $p(IJ, J)$ form a graded module octahedron (resp. braid) HC over $(P, >)$.*

A graded module octahedron (resp. braid) G which is defined as the homology of a chain complex octahedron (resp. braid) is said to be *chain complex generated*.

Proposition 3.7 ([Fra89, Prop. 3.4]). *Let $(P, >)$ be a poset. Further let $\{C(p) \mid p \in P\}$ be a collection of graded modules indexed by P and let $\Delta : \bigoplus_{p \in P} C(p) \rightarrow \bigoplus_{p \in P} C(p)$ be a lower triangular boundary map. Then the chain complexes $C^\Delta(I)$ defined by the $C(I)$ and $\Delta(I)$ together with the chain maps $i(I, IJ)$ and $p(IJ, J)$ form a chain complex octahedron (resp. braid).*

In other words, for a (C) -connection matrix for the graded module triangle G to exist, G must be a graded module octahedron (resp. braid).

3.3. Octahedra and braids are obsolete in the definition of (C) -connection matrices. When we wanted to implement the definition of connection and C -connection matrices in the Maple package `conley`, we discovered that one can completely avoid introducing the notions of chain complex octahedra (resp. braids) and of graded module octahedra (resp. braids):

First notice that for C^Δ to form a chain complex braid, the matrix Δ must only satisfy

the conditions of Proposition 3.7. By Proposition 3.6 HC^Δ is then a graded module braid. In other words, the only conditions on Δ for HC^Δ to be a graded module braid are those of Proposition 3.7:

- a) Δ is lower triangular.
- b) Δ is a boundary map.

Furthermore Remark 3.4 says that for testing the isomorphism of HC^Δ and G one only needs the isomorphism of the underlying triangles, provided that:

- (*) For each interval I there is only one isomorphism $\theta(I) : HC^\Delta(I) \rightarrow G(I)$ entering in all the commutative diagrams in (1) (resp. (2)).

3.4. FRANZOSA'S EXISTENCE RESULTS FOR C -CONNECTION MATRICES. Now that we have collected the necessary notions we can state the main existence result for C -connection matrices:

Theorem 3.8 ([Fra89], Theorem 3.8²). *Let G be a chain complex generated graded module octahedron, $C = \{C(p) \mid p \in P\}$ a collection of free graded modules and $\delta = \{\delta(p) : C(p) \rightarrow C(p) \mid p \in P\}$ a collection of boundary maps, such that the homology of $(C(p), \delta(p))$ coincides with $G(p)$. Then there exists a C -connection matrix Δ for G with $\Delta_{p,p} = \delta(p)$.*

The converse is established by Propositions 3.7 and 3.6.

Corollary 3.9. *If G is a chain complex generated graded module octahedron and $G(p)$ is free for all $p \in P$, then there exists a connection matrix Δ for G .*

Proof. Set $C(p) = G(p)$ with zero differential. □

4. SYMMETRIC CONNECTION MATRICES

Let $(P, >)$ be a poset and Γ be a subgroup of the automorphism group $\text{Aut}(P, >)$, i.e. $q > p$ implies $\sigma q > \sigma p$ for all $\sigma \in \Gamma$. The action of Γ on P induces an action on $\mathcal{J}_n(P, >)$ for all $n \in \mathbb{N}$.

Let $C = \{C(p) \mid p \in P\}$ be a collection of graded modules indexed by P . An action of Γ on C is given by specifying for each $\sigma \in \Gamma$ and each $p \in P$ an isomorphism, which we simply denote by $\psi(\sigma) : C(p) \rightarrow C(\sigma p)$, such that $\psi(\sigma\tau) = \psi(\sigma)\psi(\tau)$ for all $\sigma, \tau \in \Gamma$ and $\psi(\text{id}_P) = \text{id}$.

We say a linear map $\Delta : \bigoplus_{p \in P} C(p) \rightarrow \bigoplus_{p \in P} C(p)$ is Γ -symmetric, if for all $p, q \in P$ and $\sigma \in \Gamma$ the following diagram commutes:

$$\begin{array}{ccc} C(q) & \xrightarrow{\Delta_{q,p}} & C(p) \\ \psi(\sigma) \downarrow & & \downarrow \psi(\sigma) \\ C(\sigma q) & \xrightarrow{\Delta_{\sigma q, \sigma p}} & C(\sigma p) \end{array}$$

²A nice coincidence.

Let C be a chain complex triangle. An action of Γ on C is given by specifying for each $\sigma \in \Gamma$ and each $I \in \mathcal{J}(P, >)$ an isomorphism, which we simply denote by $\psi(\sigma) : C(I) \rightarrow C(\sigma I)$, such that

- a) $\psi(\sigma\tau) = \psi(\sigma)\psi(\tau)$ for all $\sigma, \tau \in \Gamma$ and $\psi(\text{id}_P) = \text{id}$.
- b) For all $I \in \mathcal{J}(P, >)$ and all $\sigma \in \Gamma$ the following diagram commutes:

$$\begin{array}{ccc} C(I) & \xrightarrow{\partial(I)} & C(I) \\ \psi(\sigma) \downarrow & & \downarrow \psi(\sigma) \\ C(\sigma I) & \xrightarrow{\partial(\sigma I)} & C(\sigma I) \end{array}$$

- c) For all $(I, J) \in \mathcal{J}_2(P, >)$ and all $\sigma \in \Gamma$ the following two diagrams commute:

$$\begin{array}{ccccc} C(I) & \xrightarrow{i(I,IJ)} & C(IJ) & & C(IJ) & \xrightarrow{p(IJ,J)} & C(J) \\ \psi(\sigma) \downarrow & & \downarrow \psi(\sigma) & , & \psi(\sigma) \downarrow & & \downarrow \psi(\sigma) \\ C(\sigma I) & \xrightarrow{i(\sigma I, \sigma IJ)} & C(\sigma IJ) & & C(\sigma IJ) & \xrightarrow{p(\sigma IJ, \sigma J)} & C(\sigma J) \end{array}$$

Let G be a graded module triangle. An action of Γ on G is given by specifying for each $\sigma \in \Gamma$ and each $I \in \mathcal{J}(P, >)$ an isomorphism, which we simply denote by $\psi(\sigma) : G(I) \rightarrow G(\sigma I)$, such that

- a) $\psi(\sigma\tau) = \psi(\sigma)\psi(\tau)$ for all $\sigma, \tau \in \Gamma$ and $\psi(\text{id}_P) = \text{id}$.
- b) For all $(I, J) \in \mathcal{J}_2(P, >)$ and all $\sigma \in \Gamma$ the following three diagrams commute:

$$\begin{array}{ccccccc} G(I) & \xrightarrow{i(I,IJ)} & G(IJ) & \xrightarrow{p(IJ,J)} & G(J) & \xrightarrow{\partial(J,I)} & G(I) \\ \psi(\sigma) \downarrow & & \downarrow \psi(\sigma) & , & \psi(\sigma) \downarrow & & \downarrow \psi(\sigma) \\ G(\sigma I) & \xrightarrow{i(\sigma I, \sigma IJ)} & G(\sigma IJ) & \xrightarrow{p(\sigma IJ, \sigma J)} & G(\sigma J) & \xrightarrow{\partial(\sigma J, \sigma I)} & G(\sigma I) \end{array}$$

Note that the existence of an action of non-trivial $\Gamma \leq \text{Aut}(P, >)$ is a non-trivial condition on C , resp. G .

Remark 4.1 (Symmetric C -connection matrices). Let Γ as above act on a collection $C = \{C(p) \mid p \in P\}$ of graded modules and let $\Delta : \bigoplus_{p \in P} C(p) \rightarrow \bigoplus_{p \in P} C(p)$ be a Γ -symmetric lower triangular boundary map. Then Γ acts on C^Δ via $\psi(\sigma) : C(I) \rightarrow C(\sigma I)$ given by the direct sum of the $\psi(\sigma) : C(p) \rightarrow C(\sigma p)$, for all $p \in I$. It further acts on HC^Δ in an obvious way. Hence, for such a Γ -symmetric Δ to be a C -connection matrix of a graded module octahedron G , Γ must act on G .

More generally, if Γ acts on a chain complex octahedron, it acts automatically on its homology graded module octahedron.

5. C -CONNECTION MATRICES OF A MORSE DECOMPOSITION

An inexhaustible reservoir of chain complex generated graded module octahedra are the homology graded module octahedra of a Morse decomposition of a dynamical system. This was proved in [Fra86]. See also [Fra88, Fra89, Mis95, MM02] for further details, results and applications.

Let X be a locally compact metric space. The object of study is a *flow* $\varphi : \mathbb{R} \times X \rightarrow X$, i.e. a continuous map $\mathbb{R} \times X \rightarrow X$ which satisfies $\varphi(0, x) = x$ and $\varphi(s, \varphi(t, x)) = \varphi(s+t, x)$ for all $x \in X$ and $s, t \in \mathbb{R}$.

For a subset $Y \subset X$ define

$$\text{Inv}(Y) := \text{Inv}(Y, \varphi) := \{x \in Y \mid \varphi(\mathbb{R}, x) \subset Y\} \subset Y,$$

the *invariant subset* of Y .

5.1. Homology Conley index. A subset $S \subset X$ is *invariant* under the flow φ , if $S = \text{Inv}(S)$. It is an *isolated invariant set* if there exists a *compact* set $Y \subset X$ (an *isolating neighborhood*) such that

$$S = \text{Inv}(Y) \subset Y^\circ,$$

where Y° denotes the interior of Y .

Let M be an isolated invariant set [Con78]. A pair of compact sets (N, L) with $L \subset N$ is called an *index pair* for M if

- a) $\overline{N \setminus L}$ is an *isolating neighborhood* of M .
- b) L is *positively invariant*, i.e. $\varphi([0, t], x) \subset L$ for all $x \in L$ satisfying $\varphi([0, t], x) \subset N$.
- c) L is an *exit set* for N , i.e. for all $x \in N$ and all $t_1 > 0$ such that $\varphi(t_1, x) \notin N$, there exists a $t_0 \in [0, t_1]$ for which $\varphi([0, t_0], x) \subset N$ and $\varphi(t_0, x) \in L$.

Every isolated invariant set M has an index pair (N, L) and the *homology Conley index* of M is defined by

$$CH(M) := H(N, L).$$

$CH(M)$ is as such a graded module.

5.2. Morse decomposition. For a subset $Y \subset X$ the ω -*limit set* of Y is

$$\omega(Y) := \bigcap_{t>0} \overline{\varphi([t, \infty), Y)},$$

while the α -*limit set* of Y is

$$\alpha(Y) := \bigcap_{t>0} \overline{\varphi((-\infty, -t), Y)}.$$

For two subsets $Y_1, Y_2 \subset X$ define the *set of connecting orbits*

$$\text{Con}(Y_1, Y_2) := \{x \in X \mid \alpha(x) \subset Y_1 \text{ and } \omega(x) \subset Y_2\}.$$

Let S be an isolated invariant set and $(P, >)$ be a poset. A finite collection

$$\mathcal{M}(S) = \{M(p) \mid p \in P\}$$

of disjoint isolated invariant subsets of S is called a *Morse decomposition* if there exists a strict partial order $>$ on P , such that for every $x \in S \setminus \bigcup_{p \in P} M(p)$ there exists $p, q \in P$, such that $q > p$ and $x \in \text{Con}(M(q), M(p))$.

The sets $M(p)$ are called *Morse sets*. A partial order on P satisfying this property is said to be *admissible*.

There is a partial order $>_\varphi$ induced by the flow, *generated* by the relations $q >_\varphi p$ whenever $\text{Con}(M(q), M(p)) \neq \emptyset$. This so called flow-induced order is a subset of every admissible order, and in this sense minimal. Normally this order is not known and one is content with a coarser order. If, for example, an (energy) function E is known with $E(x) > E(\varphi(t, x))$ for all $t > 0$, then defining the partial order $>_E$ by

$$q >_E p, \text{ iff } E(y) > E(x) \text{ for all } y \in M(q) \text{ and } x \in M(p),$$

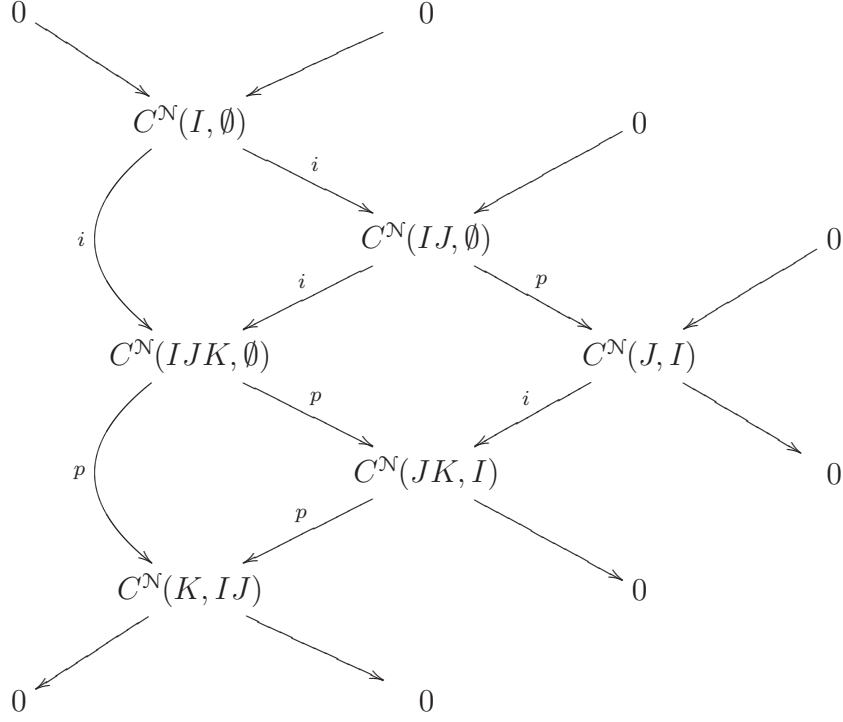
yields an admissible order.

For an interval I define the Morse set

$$M(I) := \bigcup_{p \in I} M(p) \cup \bigcup_{p, q \in I} \text{Con}(M(q), M(p)).$$

$M(I)$ is again an isolated invariant set. If $(I, J) \in \mathcal{J}_2(P, >)$, then $(M(I), M(J))$ is an attractor-repeller pair in $M(IJ)$.

Remark 5.1. In dynamical system theory FRANZOSA discovered in [Fra86] a natural chain complex octahedron, the so called chain complex octahedron $C^{\mathcal{N}}$ defined by a so called *index filtration* \mathcal{N} of a Morse decomposition. Its homology $HC^{\mathcal{N}}$ is by Proposition 3.6 a (chain complex generated) graded module octahedron:



where $C^{\mathcal{N}}(A, B) := C(\mathcal{N}(QAB), \mathcal{N}(QB))$ for A, B one of the above pairs of intervals and Q is a certain interval defined by $(I, J, K) \in \mathcal{J}_3(P, >)$. For the reasons mentioned in Remark 6.1 below we won't go into the details of this construction. The only thing one has to know is that $C(\mathcal{N}(QAB), \mathcal{N}(QB))$ is the relative singular (or simplicial, ...) chain complex³ of the pair of compact topological spaces $(\mathcal{N}(QAB), \mathcal{N}(QB))$, which is an index pair for the Morse set $M(A)$ (independent of B and $Q!$), i.e. its relative homology being the homological Conley index $CH(M(A))$ of $M(A)$.

Definition 5.2 ((C)-Connection matrix of a Morse decomposition). Given a Morse decomposition with an index filtration \mathcal{N} . A (C)-connection matrix of the Morse decomposition is a (C)-connection matrix of $HC^{\mathcal{N}}$. One can show that the definition is independent of the index filtration.

At this point we want to emphasize that in order to comply with Definition 2.7, one needs to correctly identify the $CH(M(A))$ arising as the homology of the different, but quasi-isomorphic complexes coming from the topological setup.

Remark 5.1 states that $G = HC^{\mathcal{N}}$ is automatically a chain complex generated graded module octahedron. In other words, in the context of (C)-connection matrices of Morse decompositions, one never needs to check the octahedron (resp. braid) condition. Cf. [Fra86, p. 574, below Theorem 3.8].

³In talking about the chain complex braid we use the fact that the quasi-isomorphism type of $C^{\mathcal{N}}(A, B)$ neither depends on B nor on Q .

It remains to mention why the theory of (C) -connection matrices is relevant in the context of Morse decompositions. There is a link between existence of connecting orbits and non-triviality of entries of connection matrices. For details see for example [MM02, 4.1.1].

6. THE Maple PACKAGE conley

Although the theory of C -connection matrices is a purely algebraic theory, we've chosen `conley` as the name of our Maple package in honor of CHARLES CONLEY, since his index theory [Con78] is a natural field for applying these ideas. As mentioned above, the case $G = HC^{\mathcal{N}}$ is of special interest to applications.

In Conley index theory one uses as much topological data as possible given by (the index filtration of) the Morse decomposition to restrict the possible connection matrices Δ for $HC^{\mathcal{N}}$.

Remark 6.1 (Even less is used in practice). Typically, the only data that are given are the isomorphism types of the homology Conley indices $CH(M(I))$ for *some* $I \in \mathcal{J}(P, >)$. Nevertheless, the minimal data required remains the isomorphism type of the homology Conley indices for *all* one element intervals.

As mentioned at the end of Remark 5.1, one needs, in order to apply the definition of a connection matrix, to identify for each interval I the homologies of different index pairs for $M(I)$ inside the index filtration, which are all isomorphic to $CH(M(I))$. Since this is an extremely demanding task, and up to our knowledge not addressed yet, one cannot make use of condition (*).

To be able to make use of the commutativity of the squares in (1) (resp. (2)) one needs the induced maps coming from the index triples for each pair of adjacent intervals, a data which is rarely provided.

Summing up, in the majority of non-trivial examples, the data of the index filtration for the Morse decomposition are up to our knowledge not known completely. In particular one can neither make use of condition (*) nor even of the commutativity of the squares in (1) (resp. (2)) to impose further restrictions on Δ . The only restriction left on Δ is the abstract isomorphism between $HC^{\Delta}(I)$ and $CH(M(I))$, in case the isomorphism type of the latter is known. Due to this lack of topological data the package `conley` only imposes the last mentioned restriction on Δ . If at some point in the future such data are provided, `conley` can easily be extended to make use of them.

7. EXAMPLES

In each of the four examples in this section the set of all connection matrices which are compatible with the given data of the flow under consideration are computed using the Maple package `conley`. As `conley` is based on `homalg`, the computations on the level of homological algebra are carried out by `homalg`. Since in the following examples we compute with modules over a principal ideal ring, the so-called ring package `PIR` is used to do the ring arithmetics for `homalg` [BR].

The Γ -symmetry gives more relations:

```
> Orbits(V4C2,rel_);
> rel:=map(op,%);
      rel := [[1, 0], [3, 0], [1, 5], [3, 5], [0, 2], [0, 4], [5, 2], [5, 4]]
> ord:=GeneratePartialOrder(P,rel);
ord := [[0, 2], [0, 4], [1, 0], [1, 2], [1, 4], [1, 5], [3, 0], [3, 2], [3, 4], [3, 5], [5, 2], [5, 4]]
> I1:=GenerateIntervals(P,rel);

I1 := [[], [2], [1], [5], [0], [3], [4], [4, 5], [0, 1], [0, 3], [2, 5], [1, 3], [2, 4], [0, 5], [0, 2],
[0, 4], [1, 5], [3, 5], [0, 4, 5], [2, 4, 5], [0, 1, 5], [0, 2, 5], [0, 3, 5], [1, 3, 5],
[0, 1, 3], [0, 2, 4], [0, 3, 4, 5], [0, 1, 4, 5], [0, 2, 4, 5], [0, 1, 2, 5], [0, 1, 3, 5],
[0, 2, 3, 5], [0, 2, 3, 4, 5], [0, 1, 3, 4, 5], [0, 1, 2, 4, 5], [0, 1, 2, 3, 5],
[0, 1, 2, 3, 4, 5]]
> #I123:=GenerateAdjacentIntervals(P,rel);
```

The Γ -orbits on $> \subset P \times P$:

```
> orbits:=Orbits(V4C2,ord);
      orbits := [[[0, 2], [0, 4], [5, 2], [5, 4]], [[1, 0], [3, 0], [1, 5], [3, 5]],
[[1, 2], [3, 2], [1, 4], [3, 4]]]
```

Now we have to provide the homology Conley indices for the six equilibria:

```
> CHp:=[1,2,0,2,0,1];
      CHp := [1, 2, 0, 2, 0, 1]
```

The homology of the sphere is added as homology Conley index of $M(P)$:

```
> CH:=[op(zip((x,y)->x=y,P,CHp)),P=[1,0,1]];
      CH := [0 = 1, 1 = 2, 2 = 0, 3 = 2, 4 = 0, 5 = 1, [0, 1, 2, 3, 4, 5] = [1, 0, 1]]
```

The following computation of all connection matrices takes into account the Γ -symmetry. The optional string "full_notation" means that for the matrix $\Delta_i : \bigoplus_{p \in P} CH_i(p) \rightarrow \bigoplus_{p \in P} CH_{i-1}(p)$ a zero row is reserved for every trivial $CH_i(p)$ and a zero column is reserved for every trivial $CH_{i-1}(p)$. In the hyperbolic case this leads to $|P| \times |P|$ -square matrices:

```
> Con:=ConnectionMatrices(P,rel,CH,var,"Symmetry"=V4C2,"full_notation");
```

$$Con := \left[\left[\begin{array}{cccccc} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right], \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \right]$$

```
> nops(Con);
```

```
1
```

We conclude that there is exactly one connection matrix which is compatible with the given data. It is given by the list containing the matrices which represent the boundary maps from $\bigoplus_{p \in P} CH_1(p)$ to $\bigoplus_{p \in P} CH_0(p)$ resp. from $\bigoplus_{p \in P} CH_2(p)$ to $\bigoplus_{p \in P} CH_1(p)$. Finally, we check that connecting orbits from 1 to 0 resp. from 0 to 2 exist:

```
> map(ConnectionSubmatrix,Con,P,[1,0],CHp,var);
```

```
[[[ 0 ], [ 1 ]]]
```

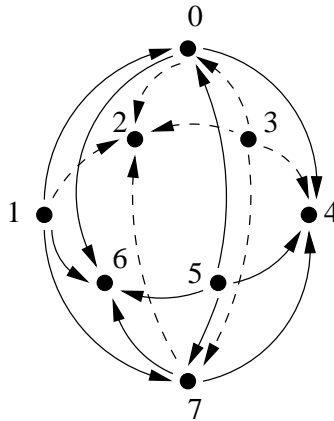
```
> map(ConnectionSubmatrix,Con,P,[0,2],CHp,var);
```

```
[[[ 1 ], [ 0 ]]]
```

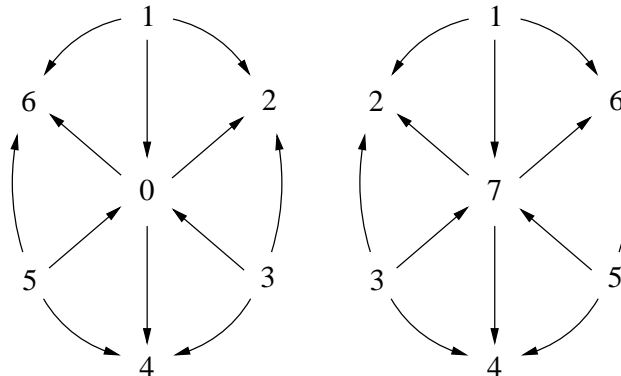
Since the format of the connection matrices may change in the future, we recommend the user to use the above way of accessing the entries of the connection matrices.

7.2. A flow with $D_6 \times C_2 \cong D_{12}$ -symmetry.

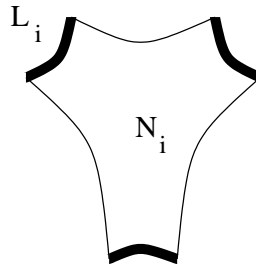
Example 7.2. Here we use `conley` to compute the connection matrices of a D_{12} -symmetric flow on the 2-sphere with six hyperbolic and two non-hyperbolic equilibria. They are enumerated as shown in the following figure:



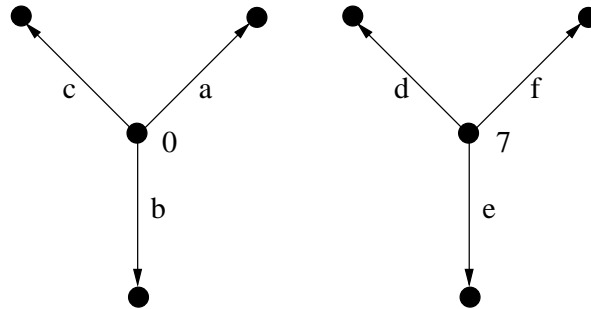
The following two figures give a simple view of the upper resp. lower hemisphere:



Around 0 resp. 7 an isolating neighborhood looks like:



(N_i, L_i) is an index pair for $i = 0, 7$, where L_i is the disjoint union of the three bold border regions of the isolating neighborhood N_i of i . The index pair (N_i, L_i) is homotopy equivalent to the simplicial complex:



```
> restart;
> with(conley): with(PIR): with(homalg):
> 'homalg/default' := 'PIR/homalg':
```

We set up the field of coefficients to be $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$:

```
> var := [2];
                                var := [2]

> Pvar(var);
                                ["Z/pZ", 2]
```

We define the set P of equilibria and the generators for $\Gamma := D_{12}$:

```
> P := [0, 1, 2, 3, 4, 5, 6, 7];
                                P := [0, 1, 2, 3, 4, 5, 6, 7]

> D6C2 := [ [[1, 3, 5], [2, 4, 6]], [[3, 5], [2, 6]], [[0, 7]] ];
                                D6C2 := [[[1, 3, 5], [2, 4, 6]], [[3, 5], [2, 6]], [[0, 7]]]
```

The action of the generating permutations $(1, 3, 5)(2, 4, 6)$, $(3, 5)(2, 6)$ and $(0, 7)$ on $\bigoplus_{p \in P} CH_i(p)$ for $i = 0, 1, 2$ is represented by the following matrices. More precisely, for each generator we give an equation whose left hand side is the permutation π of the set P and whose right hand side is the corresponding list $[M_0(\pi), M_1(\pi), M_2(\pi)]$ of matrices representing the action of π on the respective $\bigoplus_{p \in P} CH_i(p)$, $i = 0, 1, 2$. The dimensions of the matrices are coherent with the so called standard notation ("std_notation") explained below:

```

> id:=linalg[diag](1$3);
> rho:=matrix([[0,1,0],[0,0,1],[1,0,0]]);
> rep=[ [rho,matrix([[0,1,0,0],[1,1,0,0],[0,0,0,1],[0,0,1,1]]),rho],
> [matrix([[0,0,1],[0,1,0],[1,0,0]]),
> matrix([[1,1,0,0],[0,1,0,0],[0,0,1,1],[0,0,0,1]]),
> matrix([[1,0,0],[0,0,1],[0,1,0]])],
> [id,matrix([[0,0,1,0],[0,0,0,1],[1,0,0,0],[0,1,0,0]]),id]]:

```

$$id := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rho := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

```

> D6C2_rep:=zip((a,b)->a=b,D6C2,rep);

```

$$D6C2_rep := \left[\left[[1, 3, 5], [2, 4, 6] \right] = \left[\rho, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \rho \right], \right.$$

$$\left. \left[[3, 5], [2, 6] \right] = \left[\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right], \right.$$

$$\left. \left[[0, 7] \right] = \left[id, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, id \right]$$

A simple calculation shows that

$$CH_1(0) := H_1(N_0, L_0) \cong \mathbb{F}_2 a \oplus \mathbb{F}_2 b \oplus \mathbb{F}_2 c / \mathbb{F}_2(a + b + c) = \langle \bar{a}, \bar{b} \rangle_{\mathbb{F}_2} \cong \mathbb{F}_2^{1 \times 2}$$

resp.

$$CH_1(7) := H_1(N_7, L_7) \cong \mathbb{F}_2 d \oplus \mathbb{F}_2 e \oplus \mathbb{F}_2 f / \mathbb{F}_2(d + e + f) = \langle \bar{d}, \bar{e} \rangle_{\mathbb{F}_2} \cong \mathbb{F}_2^{1 \times 2}.$$

The \mathbb{F}_2 -basis of $\bigoplus_{p \in P} CH_1(p) = CH_1(0) \oplus CH_1(7) \cong \mathbb{F}_2^{1 \times 2} \oplus \mathbb{F}_2^{1 \times 2} \cong \mathbb{F}_2^{1 \times 4}$ which was chosen to write down the above matrices is $(\bar{a}, \bar{b}, \bar{d}, \bar{e})$.

Next we define the generating set of relations for >:

```

> rel_:=[[1,0],[0,2]];

```

$$rel_ := [[1, 0], [0, 2]]$$

The orbits of these relations under the given symmetry group are obtained as follows:

```

> Orbits(D6C2,rel_):
> rel:=map(op,%);

```

```
rel := [[1, 0], [3, 0], [1, 7], [5, 0], [3, 7], [5, 7], [0, 2], [0, 4], [0, 6], [7, 2], [7, 4], [7, 6]]
> ord:=GeneratePartialOrder(P,rel);
```

```
ord := [[0, 2], [0, 4], [0, 6], [1, 0], [1, 2], [1, 4], [1, 6], [1, 7], [3, 0], [3, 2], [3, 4], [3, 6],
[3, 7], [5, 0], [5, 2], [5, 4], [5, 6], [5, 7], [7, 2], [7, 4], [7, 6]]
```

We list here all intervals of points in P which are admitted by the given ordering >:

```
> I1:=GenerateIntervals(P,rel);

I1 := [[], [2], [1], [0], [7], [3], [4], [5], [6], [3, 5], [2, 6], [0, 7], [0, 2], [1, 7], [3, 7], [5, 7],
[0, 4], [0, 6], [0, 1], [0, 3], [0, 5], [1, 3], [1, 5], [2, 4], [2, 7], [4, 6], [4, 7], [6, 7],
[1, 3, 5], [2, 4, 6], [0, 6, 7], [2, 6, 7], [4, 6, 7], [0, 1, 7], [0, 2, 7], [0, 3, 7], [1, 3, 7],
[0, 4, 7], [2, 4, 7], [0, 5, 7], [1, 5, 7], [3, 5, 7], [0, 1, 3], [0, 2, 4], [0, 1, 5], [0, 3, 5],
[0, 2, 6], [0, 4, 6], [0, 5, 6, 7], [0, 1, 6, 7], [0, 2, 6, 7], [0, 3, 6, 7], [0, 4, 6, 7],
[2, 4, 6, 7], [0, 1, 2, 7], [0, 1, 3, 7], [0, 2, 3, 7], [0, 1, 4, 7], [0, 2, 4, 7], [0, 3, 4, 7],
[0, 1, 5, 7], [0, 2, 5, 7], [0, 3, 5, 7], [1, 3, 5, 7], [0, 4, 5, 7], [0, 1, 3, 5], [0, 2, 4, 6],
[0, 4, 5, 6, 7], [0, 1, 5, 6, 7], [0, 2, 5, 6, 7], [0, 3, 5, 6, 7], [0, 1, 2, 6, 7],
[0, 1, 3, 6, 7], [0, 2, 3, 6, 7], [0, 1, 4, 6, 7], [0, 2, 4, 6, 7], [0, 3, 4, 6, 7],
[0, 1, 2, 3, 7], [0, 1, 2, 4, 7], [0, 1, 3, 4, 7], [0, 2, 3, 4, 7], [0, 1, 2, 5, 7],
[0, 1, 3, 5, 7], [0, 2, 3, 5, 7], [0, 1, 4, 5, 7], [0, 2, 4, 5, 7], [0, 3, 4, 5, 7],
[0, 3, 4, 5, 6, 7], [0, 1, 4, 5, 6, 7], [0, 2, 4, 5, 6, 7], [0, 1, 2, 5, 6, 7], [0, 1, 3, 5, 6, 7],
[0, 2, 3, 5, 6, 7], [0, 1, 2, 3, 6, 7], [0, 1, 2, 4, 6, 7], [0, 1, 3, 4, 6, 7], [0, 2, 3, 4, 6, 7],
[0, 1, 2, 3, 4, 7], [0, 1, 2, 3, 5, 7], [0, 1, 2, 4, 5, 7], [0, 1, 3, 4, 5, 7], [0, 2, 3, 4, 5, 7],
[0, 2, 3, 4, 5, 6, 7], [0, 1, 3, 4, 5, 6, 7], [0, 1, 2, 4, 5, 6, 7], [0, 1, 2, 3, 5, 6, 7],
[0, 1, 2, 3, 4, 6, 7], [0, 1, 2, 3, 4, 5, 7], [0, 1, 2, 3, 4, 5, 6, 7]]

> #I123:=GenerateAdjacentIntervals(P,rel);
> orbits:=Orbits(D6C2,ord);
```

```
orbits := [[[0, 2], [0, 4], [0, 6], [7, 2], [7, 4], [7, 6]],
[[1, 0], [3, 0], [1, 7], [5, 0], [3, 7], [5, 7]], [[1, 2], [3, 4], [1, 6], [5, 6], [5, 4], [3, 2]],
[[1, 4], [3, 6], [5, 2]]]
```

Next we provide the homology Conley indices. Note that since 0 and 7 are non-hyperbolic, their homology is not of rank 1.

```
> CHp:=[[0,2],2,0,2,0,2,0,[0,2]];
      CHp := [[0, 2], 2, 0, 2, 0, 2, 0, [0, 2]]
> CH:=[op(zip((x,y)->x=y,P,CHp)),P=[1,0,1]];
      CH := [0 = [0, 2], 1 = 2, 2 = 0, 3 = 2, 4 = 0, 5 = 2, 6 = 0, 7 = [0, 2],
[0, 1, 2, 3, 4, 5, 6, 7] = [1, 0, 1]]
```

Now we compute all connection matrices for the given homology Conley indices respecting the symmetry group Γ generated by D6C2, where the action of Γ on $\bigoplus_{p \in P} CH_i(p)$ for $i = 0, 1, 2$ by the matrices defined above is taken into account. The optional string "std_notation" means that the matrix $\Delta_i : \bigoplus_{p \in P} CH_i(p) \rightarrow \bigoplus_{p \in P} CH_{i-1}(p)$ is a $d_i \times d_{i-1}$ -matrix, where $d_j := \dim_{\mathbb{F}_2} \bigoplus_{p \in P} CH_j(p)$:

> `Con:=ConnectionMatrices(P,rel,CH,var,"Symmetry"=D6C2_rep,"std_notation");`

$$Con := \left[\left[\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \right] \right]$$

We conclude that there is exactly one connection matrix which is compatible with the given data. It is given by the list containing the matrices which represent the boundary maps from $\bigoplus_{p \in P} CH_1(p)$ to $\bigoplus_{p \in P} CH_0(p)$ resp. from $\bigoplus_{p \in P} CH_2(p)$ to $\bigoplus_{p \in P} CH_1(p)$. Here we extract the information about connecting orbits between points in P which is provided by the computed connection matrix:

> `ConnectionSubmatrix(Con[1],P,[0,2],CHp,var),ConnectionSubmatrix(Con[1],P,[0,4],CHp,var),ConnectionSubmatrix(Con[1],P,[0,6],CHp,var);`
 > `ConnectionSubmatrix(Con[1],P,[7,2],CHp,var),ConnectionSubmatrix(Con[1],P,[7,4],CHp,var),ConnectionSubmatrix(Con[1],P,[7,6],CHp,var);`

$$\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix}, [0], \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [0], \begin{bmatrix} 1 \\ 1 \end{bmatrix}, [0] \right]$$

$$\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix}, [0], \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [0], \begin{bmatrix} 1 \\ 1 \end{bmatrix}, [0] \right]$$

Hence, there are connecting orbits from 0 to 2, to 4, to 6, and from 7 to 2, to 4, and to 6.

> `ConnectionSubmatrix(Con[1],P,[1,0],CHp,var),ConnectionSubmatrix(Con[1],P,[1,7],CHp,var);`
 > `ConnectionSubmatrix(Con[1],P,[3,0],CHp,var),ConnectionSubmatrix(Con[1],P,[3,7],CHp,var);`
 > `ConnectionSubmatrix(Con[1],P,[5,0],CHp,var),ConnectionSubmatrix(Con[1],P,[5,7],CHp,var);`

$$\left[[0], [0 \ 1], [[0], [0 \ 1]] \right]$$

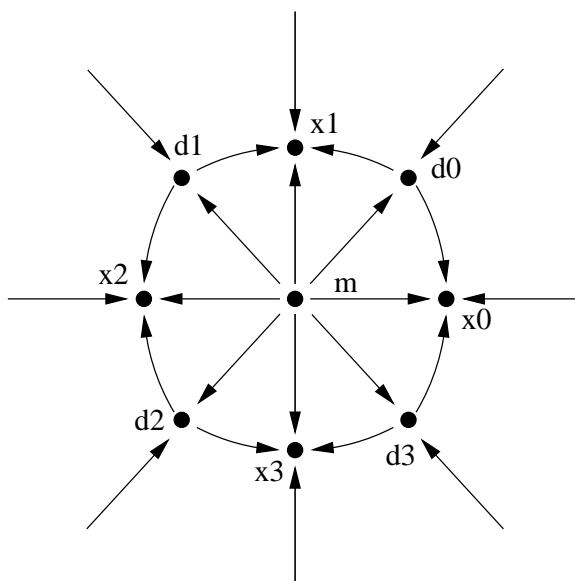
$$\left[[0], [1 \ 1], [[0], [1 \ 1]] \right]$$

$$\left[[0], [1 \ 0], [[0], [1 \ 0]] \right]$$

Similarly, there are connecting orbits from 1 to 0 and 7, from 3 to 0 and 7, and from 5 to 0 and 7.

7.3. The Cahn-Hilliard equation.

Example 7.3. In this example we consider a D_8 -symmetric flow with nine hyperbolic equilibria studied in [MPMW, 2.2, 4.2]. We stick to the notation of the equilibria of that reference:



```

> with(conley): with(PIR): with(homalg):
> 'homalg/default':='PIR/homalg':
> var:=[2];
                                var := [2]
> Pvar(var);
                                ["Z/pZ", 2]
> P:=[x0,x1,x2,x3,d0,d1,d2,d3,m];
                                P := [x0, x1, x2, x3, d0, d1, d2, d3, m]
> CHp:=[0,0,0,0,1,1,1,1,2];
                                CHp := [0, 0, 0, 0, 1, 1, 1, 1, 2]
> rel:=DefinePartialOrderByPotential(P,CHp,P);
rel := [[d0, x0], [d1, x0], [d2, x0], [d3, x0], [m, x0], [d0, x1], [d1, x1], [d2, x1], [d3, x1],
[m, x1], [d0, x2], [d1, x2], [d2, x2], [d3, x2], [m, x2], [d0, x3], [d1, x3], [d2, x3],
[d3, x3], [m, x3], [m, d0], [m, d1], [m, d2], [m, d3]]
> ord:=GeneratePartialOrder(P,rel);
ord := [[d0, x0], [d0, x1], [d0, x2], [d0, x3], [d1, x0], [d1, x1], [d1, x2], [d1, x3], [d2, x0],
[d2, x1], [d2, x2], [d2, x3], [d3, x0], [d3, x1], [d3, x2], [d3, x3], [m, x0], [m, x1],
[m, x2], [m, x3], [m, d0], [m, d1], [m, d2], [m, d3]]
> D8:=[ [[x0,x1,x2,x3], [d0,d1,d2,d3]], [[x1,x3], [d0,d3], [d1,d2]] ];
D8 := [[[x0, x1, x2, x3], [d0, d1, d2, d3]], [[x1, x3], [d0, d3], [d1, d2]]]
> CH:=[op(zip((x,y)->x=y,P,CHp)),P=0];

```

```

CH := [x0 = 0, x1 = 0, x2 = 0, x3 = 0, d0 = 1, d1 = 1, d2 = 1, d3 = 1, m = 2,
[x0, x1, x2, x3, d0, d1, d2, d3, m] = 0]
> Con:=ConnectionMatrices(P,rel,CH,var,"Symmetry"=D8,"full_notation");

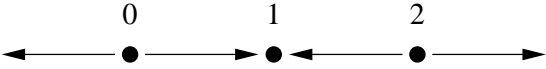
```

$$Con := \left(\left[\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{array} \right] \right), \\
\left(\left[\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{array} \right] \right)$$

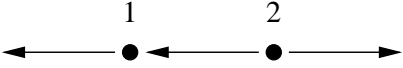
We obtain two connection matrices. The full flow diagram above would give a unique D_8 -symmetric connection matrix, but this diagram was derived in [MPMW] using dynamical arguments. Here we demonstrated that the lack of the dynamical arguments produces ambiguities. This is the point of departure in a joint work in progress with MAIER-PAAPE.

7.4. An example for a C -connection matrix.

Example 7.4. Here we demonstrate on a tiny example the advantage of a C -connection matrix over a connection matrix. The aim is to look at the two equilibria 0 and 1 of the flow visualized in



as a black box. Then the flow can also be depicted as follows:



Here we named the black box equilibrium again by 1.

```

> with(conley): with(PIR): with(homalg):
> 'homalg/default' := 'PIR/homalg':
> var:=[2];

```



```

                                var := [2]
> Pvar(var);
                                ["Z/pZ", 2]
> P:=[0,1,2];
                                P := [0, 1, 2]

```

The indices:

```

> CHp:=[1,0,1];
                                CHp := [1, 0, 1]
> rel:=[[0,1],[2,1]];
                                rel := [[0, 1], [2, 1]]
> ord:=GeneratePartialOrder(P,rel);
                                ord := [[0, 1], [2, 1]]
> I1:=GenerateIntervals(P,rel);
                                I1 := [[], [0], [1], [2], [0, 1], [0, 2], [1, 2], [0, 1, 2]]
> CH:=op(zip((x,y)->x=y,P,CHp)), [0,1]=[], [1,2]=[], P=1];
                                CH := [0 = 1, 1 = 0, 2 = 1, [0, 1] = [], [1, 2] = [], [0, 1, 2] = 1]
> Con:=ConnectionMatrices(P,rel,CH,var,"full_notation");
                                Con := 
$$\begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{bmatrix}$$


```

Now we consider [0,1] as one point which we call 1:

```

> Q:=[1,2];
                                Q := [1, 2]
> Cp:=[1=[1,1],2=1];
                                Cp := [1 = [1, 1], 2 = 1]
> Crel:=[[2,1]];
                                Crel := [[2, 1]]
> CCH:=[1=[],Q=1];
                                CCH := [1 = [], [1, 2] = 1]
> CCon:=CConnectionMatrices(Q,Crel,Cp,CCH,var,"full_notation");
                                CCon := 
$$\left[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right]$$

> map(ConnectionSubmatrix,CCon,Q,[1,1],map(rhs,Cp),var);
                                
$$\left[ \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix} \right]$$


```

```

> map(ConnectionSubmatrix,CCon,Q,[2,2],map(rhs,Cp),var);
      [[[ 0 ]], [[ 0 ]]]
> map(ConnectionSubmatrix,CCon,Q,[1,2],map(rhs,Cp),var);
      [[[ 0 ]], [[ 0 ]]]
> map(ConnectionSubmatrix,CCon,Q,[2,1],map(rhs,Cp),var);
      [[[ 0 ]], [[ 1 ]]]

```

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