Vessiot's Equivalence Method Applied to Linear Partial Differential Operators

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Outline

Motivation

Example

Linear partial differential operators (LPDOs) of order 2:

$$L = D_{x^1} D_{x^2} + a\, D_{x^1} + b\, D_{x^2} + c$$

under gauge transformations:

$$L \mapsto g^{-1} L g, \qquad g = g(x^1, x^2).$$

Questions:

- Equivalence: L, L' $\quad \rightsquigarrow \quad$ $g^{-1} L g = L'$?
- Invariants?

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Questions:

- Equivalence: L, L' $\quad \rightsquigarrow \quad g^{-1} L g = L'$?
- Invariants? Laplace:

$$h = c - a_{x^1} - ab, \qquad k = c - b_{x^2} - ab.$$

- Generating set of invariants?
- Larger examples?

Outline

Natural bundles

Let X be a manifold, coordinates $(x) = (x^1, \ldots, x^n)$.

- $\mathrm{Diff}_{\mathrm{loc}}(X, X)$: local diffeomorphisms $\varphi : X \to X$.
- A natural bundle is a fibre bundle

$$\pi : \mathcal{F} \to X : (x, u) \to (x)$$

  such that each $\varphi \in \mathrm{Diff}_{\mathrm{loc}}(X, X)$ continues to $\tilde{\varphi} : \mathcal{F} \to \mathcal{F}$.
- A section $\omega : X \to \mathcal{F} : (x) \mapsto (x, u = \omega(x))$ is called geometric object.

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- φ is an equivalence between ω, $\gamma : X \to \mathcal{F} \Leftrightarrow$

  $$\omega \circ \tilde{\varphi} = \gamma$$

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- $\psi : \mathcal{F} \to \mathbb{R}$ is an invariant if $\psi \circ \tilde{\varphi} = \psi \quad \forall \varphi \in \mathrm{Diff}_{\mathrm{loc}}(X, X)$.

Symmetry Groupoids

- The jet groupoid $\Pi_q = \Pi_q(X, X)$ has coordinates:

$$(x, y, y_x, \ldots, y_q).$$

- Like $\mathrm{Diff}_{\mathrm{loc}}(X, X)$, Π_q acts on $\mathcal{F}$.
- Symmetry groupoid $\mathcal{R}_q(\omega)$ of $\omega : X \to \mathcal{F}$ defined by:

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- Prolongation

$$0 \longrightarrow \mathcal{R}_{q+1}(\omega) \longrightarrow \Pi_{q+1} \underset{j_1(\omega)}{\overset{j_1(\Phi_\omega)}{\rightrightarrows}} J_1(\mathcal{F})$$

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Question: Is $\mathcal{R}_q^{(1)}(\omega) = \mathcal{R}_q(\omega)$?

Integrability Conditions

Theorem

$\pi_q^{q+1} : \mathcal{R}_{q+1}(\omega) \to \mathcal{R}_q(\omega)$ is surjective if and only if there is a section $c : \mathcal{F} \to \mathcal{F}_1$:

- *c is equivariant:*

$$c(af_q) = c(a)f_q \qquad \forall f_q \in \Pi_q,\ a \in \mathcal{F}$$

- *c fulfills the Vessiot structure equations:*

$$(I \circ j_1)(\omega) = c(\omega).$$

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If in addition the symbol of $\mathcal{R}_q(\omega)$ is 2-acyclic, $\mathcal{R}_q(\omega)$ is integrable, i. e.:

$$\pi_{q+r}^{q+r+s} : \mathcal{R}_{q+r+s}(\omega) \to \mathcal{R}_{q+r}(\omega) \qquad \forall r, s \in \mathbb{N}$$

are all surjective.

Equivalence

Theorem

Two geometric objects ω and γ on $\mathcal{F}$ are equivalent if:

- *All invariants ψ on $\mathcal{F}$ coincide for some $x, y \in X$:*

$$\psi(\omega(y)) = \psi(\gamma(x)),$$

- *$\mathcal{R}_q(\omega)$, $\mathcal{R}_q(\gamma)$ are integrable with $c : \mathcal{F} \to \mathcal{F}_1$:*

$$\begin{aligned} (I \circ j_1)(\omega) &= c(\omega), \\ (I \circ j_1)(\gamma) &= c(\gamma). \end{aligned}$$

Example

In the introductory example (ω, Ω) is equivalent to $(\bar{\omega}, \bar{\Omega})$ if:

$$d\omega = c_1 \Omega, \quad d\bar{\omega} = c_1 \bar{\Omega}$$

Outline

Groupoids Θ_q of Gauge Transformations

- A simple observation for LPDOs:

$$\left(g^{-1}Lg\right)u(x) = g^{-1}L\left(gu(x)\right)$$

$\rightsquigarrow$ Find a groupoid containing only transformations:

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- On $Y = X \times \mathbb{R}$, coordinates (x,u), (y,v) define $\Theta_q \leq \Pi_q(Y,Y)$:

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- All previous slides remain true if Π_q is restricted to Θ_q:

$$0 \longrightarrow \mathcal{R}_q(\omega) \longrightarrow \Theta_q \underset{\omega}{\overset{\Phi_\omega}{\rightrightarrows}} \mathcal{F}$$

- Natural Θ_q-bundles: Θ_q-action on $\mathcal{F}$.

Laplace Example – Natural Bundle

- Construct a natural Θ_q-bundle for:

$$L = D_{x^1} D_{x^2} + a D_{x^1} + b D_{x^2} + c.$$

- Gauge transformation:

$$\begin{aligned} g^{-1} L g &= D_{x^1} D_{x^2} + \left(\frac{g_{x^2}}{g} + a\right) D_{x^1} + \left(\frac{g_{x^1}}{g} + b\right) D_{x^2} \\ &\quad + \frac{g_{x^1 x^2}}{g} + a\,\frac{g_{x^1}}{g} + b\,\frac{g_{x^2}}{g} + c \end{aligned}$$

- Reminder: Θ_q : $\quad y = x, \quad u\, v_u = v$.
  $\Rightarrow v(x,u) = g(x) u$.
- Natural Θ_q-bundle $\mathcal{F} = Y \times \mathbb{R}^3$, coordinates (x, u, a, b, c):

$$\begin{aligned} a &= \left(\frac{v_{x^2}}{v} + \hat{a}\right), \\ b &= \left(\frac{v_{x^1}}{v} + \hat{b}\right), \\ c &= \frac{v_{x^1 x^2}}{v} + \hat{a}\,\frac{v_{x^1}}{v} + \hat{b}\,\frac{v_{x^2}}{v} + \hat{c}. \end{aligned}$$

Laplace Example – Prolongation

$$L = D_{x^1} D_{x^2} + a D_{x^1} + b D_{x^2} + c$$

- Θ_q-action on $\mathcal{F} = Y \times \mathbb{R}^3$:

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- Coordinates of $J_1(\mathcal{F})$:

  $$a_{x^1}, \quad a_{x^2}, \quad a_u, \quad b_{x^1}, \quad b_{x^2}, \quad b_u, \quad c_{x^1}, \quad c_{x^2}, \quad c_u.$$

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- Coordinates of $J_{1,X}(\mathcal{F})$:

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Theorem

If the Θ_q-action on $\mathcal{F}$ depends on v_{x^μ}/v only then

- *Sections $\omega(x)$ of $\mathcal{F}$ are well-defined.*
- *$j_1(\omega)(x)$ restricts to the Θ_q-subbundle $J_{1,X}(\mathcal{F}) \subseteq J_1(\mathcal{F})$.*

Invariants on Θ_q-bundles

- Θ_q is defined by:

$$y = x, \quad u\, v_u = v.$$

- Invariant $\psi : J_{r,X}(\mathcal{F}) \to \mathbb{R}$.
- Derivatives $D_{x^i}\psi$ are invariants on $J_{r+1,X}(\mathcal{F})$.

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- If all coordinates of $\mathcal{F}_1$ are invariants,

$$\mathcal{F}_1 \cong \mathcal{F} \times \mathbb{R}^k$$

  $\Rightarrow$ the invariants on $\mathcal{F}_1$ are a generating set.
- Next bundle of integrability conditions $\mathcal{F}_2$:

$$\mathcal{F}_2 \cong \mathcal{F}_1 \times \mathbb{R}^{nk}$$

Laplace Example – Vessiot Structure Equations

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- Coordinates of $\mathcal{F}_1$:

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- Better coordinates of $\mathcal{F}_1$:

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- No equivariant sections $\mathcal{F} \to \mathcal{F}_1$.
- Vessiot structure equations on $\mathcal{F}_2 \cong \mathcal{F}_1 \times \mathbb{R}^4$:

$$h_{x^1} = c_1(h,k), \;\; h_{x^2} = c_2(h,k), \;\; k_{x^1} = c_3(h,k), \;\; k_{x^2} = c_4(h,k).$$

$\Rightarrow \{h, k\}$ is a generating set of invariants.

Example from Shemyakova & Winkler [SW07]

Example

Third order LPDOs under gauge transformations:

$$(D_x + qD_y)D_{xy} + a_{20}D_{xx} + a_{11}D_{xy} + a_{02}D_{yy} + a_{10}D_x + a_{01}D_y + a_{00}$$

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Equivalence conditions: $\rightsquigarrow$ second order invariants.

Order	Σ	Invariants
0	2	$q, \quad I_1 = 2a_{20}q^2 - a_{11}q + 2a_{02}$
1	8	q_x, q_y, I_x^1, I_y^1 4 new invariants
2	15	14 = 6 + 8 old, 1 new
3	21	21 old

Generating set from [SW07] is very compact!

The end.

Thanks!

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