Symplectic PBW Tableaux and Degenerate Relations

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1. Original and PBW degenerate complete symplectic flag varieties.

General description. Fix a simple Lie group G over $\mathbb{C}$, let $\mathfrak{g}$ be its Lie algebra and P^+ the set of regular, dominant, integral weights. For $\lambda \in P^+$, let V_λ be the simple $\mathfrak{g}$-module, and $\nu_\lambda \in \mathrm{V}_\lambda$ a highest weight vector.

Definition

The flag variety $\mathcal{F}_\lambda$ is the closure of the G-orbit through a highest weight line: $\mathcal{F}_\lambda = \overline{\mathrm{G}[\nu_\lambda]} \hookrightarrow \mathbb{P}(\mathrm{V}_\lambda)$.

Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and $\mathcal{U}(\mathfrak{n}^-)$ be the universal env. algebra of $\mathfrak{n}^-$. Then $\mathrm{V}_\lambda = \mathcal{U}(\mathfrak{n}^-)\nu_\lambda$ and there exists a degree filtration

$$\mathcal{U}(\mathfrak{n}^-)_s := \mathrm{span}\{x_1 \cdots x_l : \; x_i \in \mathfrak{n}^-, \; l \leq s\}.$$

The induced filtration $F_s := \mathcal{U}(\mathfrak{n}^-)_s \nu_\lambda$ on V_λ is called the *PBW filtration*.

Let V_λ^a be the graded space associated to the filtration F_s, then

$$\mathrm{V}_\lambda^a = F_0 \oplus_{s \geq 1} F_s / F_{s-1}, \qquad F_0 = \mathbb{C}\nu_\lambda.$$

The space V_λ^a is a $\mathfrak{g}^a$-module, where $\mathfrak{g}^a = \mathfrak{b} \oplus (\mathfrak{n}^-)^a$ and $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$. Let G^a be a Lie group corresponding to $\mathfrak{g}^a$. Let ν_λ^a be the image of ν_λ in V_λ^a.

Definition

The PBW degenerate flag variety is defined to be $\mathcal{F}_\lambda^a := \overline{\mathrm{G}^a[\nu_\lambda^a]} \hookrightarrow \mathbb{P}(\mathrm{V}_\lambda^a)$.

The symplectic complete flag variety. Denote it by $\mathrm{Sp}\mathcal{F}_{2n}$. It concides with the subvariety of full flags $\mathrm{U}_1 \subset \cdots \subset \mathrm{U}_n \subset \mathbb{C}^{2n}$, $\dim \mathrm{U}_k = k$, such that each U_k is isotropic w.r.t the nondegenerate sympl. form on $\mathbb{C}^{2n}$ whose matrix is given by:

$\mathrm{M}_s := \begin{bmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{bmatrix}$, I_n is the $n \times n$ matrix with 1's on the anti-diagonal and zero's elsewhere.

- In other words, we require that each U_k is an element of the symplectic Grassmannian $\mathrm{SpGr}(k, 2n)$.

- Consider the irreducible fundamental Sp_{2n}-module V_{ω_k} of highest weight ω_k. We have $\mathrm{V}_{\omega_1} \simeq \mathbb{C}^{2n}$ and the canonical embedding,

$$\mathrm{V}_{\omega_k} \hookrightarrow \bigwedge^k \mathbb{C}^{2n}, \quad \omega_k \mapsto \mathsf{w}_1 \wedge \cdots \wedge \mathsf{w}_k.$$

- Now consider $\mathrm{U}_k \subset \mathbb{C}^{2n}$. For $i \in \{1, \ldots, n\}$, let $\bar{i} := 2n + 1 - i$. For $\mathrm{J} = (j_1 < \cdots < j_k) \subset \{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$, let $\mathsf{w}_{\mathrm{J}} := [\mathsf{w}_{j_1} \wedge \cdots \wedge \mathsf{w}_{j_k}] \in \mathbb{P}\Big(\bigwedge^k \mathbb{C}^{2n}\Big)$ and $\mathrm{X}_{\mathrm{J}} \in \mathrm{V}^*_{\omega_k}$ the Plücker coordinate.

One has: $\mathrm{SP}\mathcal{F}_{2n} \hookrightarrow \prod_{k=1}^{n} \mathrm{SPGr}(k, 2n) \hookrightarrow \prod_{k=1}^{n} \mathbb{P}(\bigwedge^k \mathbb{C}^{2n})$.
Let $\mathbb{C}[\mathrm{X_J}]$ be the polynomial ring in variables $\mathrm{X_J}$ for all J, $k = 1, \ldots, n$.

Defining ideal. Let this ideal be denoted by I, what are its generators?

- Plücker relations. These are the relations

$$R^t_{\mathrm{L,J}} := \mathrm{X_L X_J} - \sum \mathrm{X_{L'} X_{J'}}, \tag{1.1}$$

labeled by the sequences L, J; $n \geq |\mathrm{L}| \geq |\mathrm{J}| \geq 1$ and a number t; $1 \leq t \leq |\mathrm{J}|$.

Question:
Are the relations $R^t_{\mathrm{L,J}}$ enough to generate I?

- Linear relations. Consider $\mathrm{SpGr}(2,6)$. Let $\mathrm{U}_2 \in \mathrm{SpGr}(2,6)$ be the subspace of $\mathbb{C}^6$ generated by the vectors
  $u = a_{1,1}w_1 + a_{2,1}w_2 + a_{3,1}w_3 + a_{4,1}w_4 + a_{5,1}w_5 + a_{6,1}w_6$ and
  $v = a_{1,2}w_1 + a_{2,2}w_2 + a_{3,2}w_3 + a_{4,2}w_4 + a_{5,2}w_5 + a_{6,2}w_6$.
  The subspace U_2 is isotropic if and only if $u^T \mathrm{M}_s v = 0$, i.e.,

$$\begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} & a_{4,1} & a_{5,1} & a_{6,1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \\ a_{4,2} \\ a_{5,2} \\ a_{6,2} \end{pmatrix} = 0.$$

$$\Rightarrow -a_{6,1}a_{1,2} - a_{5,1}a_{2,2} - a_{4,1}a_{3,2} + a_{3,1}a_{4,2} + a_{2,1}a_{5,2} + a_{1,1}a_{6,2} = 0.$$

  We have: $\mathrm{X}_{1,6} + \mathrm{X}_{2,5} + \mathrm{X}_{3,4} = 0$ (or $\mathrm{X}_{1,\overline{1}} = -\mathrm{X}_{2,\overline{2}} - \mathrm{X}_{3,\overline{3}}$).
  Let S_{L} denote all the relations of this kind. We call these the symplectic linear relations.

Theorem (De Concini '79)

The ideal I *is generated by the relations* $R^t_{\mathrm{L},\mathrm{J}}$ *and* S_{L}. *It is a prime ideal.*

Example

Conider $\mathfrak{sp}_4$, then the ideal I for the variety $\mathrm{S}\mathrm{P}\mathcal{F}_4$ is generated by the relations:

$R^1_{(1,2),(\overline{2})} := \mathrm{X}_{1,2}\mathrm{X}_{\overline{2}} + \mathrm{X}_{2,\overline{2}}\mathrm{X}_1 - \mathrm{X}_{1,\overline{2}}\mathrm{X}_2,$

$R^1_{(1,2),(\overline{1})} := \mathrm{X}_{1,2}\mathrm{X}_{\overline{1}} + \mathrm{X}_{2,\overline{1}}\mathrm{X}_1 - \mathrm{X}_{1,\overline{1}}\mathrm{X}_2,$

$R^1_{(1,\overline{2}),(\overline{1})} := \mathrm{X}_{1,\overline{2}}\mathrm{X}_{\overline{1}} + \mathrm{X}_{\overline{2},\overline{1}}\mathrm{X}_1 - \mathrm{X}_{1,\overline{1}}\mathrm{X}_{\overline{2}},$

$R^1_{(1,2),(\overline{2},\overline{1})} := \mathrm{X}_{1,2}\mathrm{X}_{\overline{2},\overline{1}} - \mathrm{X}_{1,\overline{2}}\mathrm{X}_{2,\overline{1}} + \mathrm{X}_{1,\overline{1}}\mathrm{X}_{2,\overline{2}},$

$R^1_{(2,\overline{2}),(\overline{1})} := \mathrm{X}_{2,\overline{2}}\mathrm{X}_{\overline{1}} + \mathrm{X}_{\overline{2},\overline{1}}\mathrm{X}_2 - \mathrm{X}_{2,\overline{1}}\mathrm{X}_{\overline{2}},$

and the linear relation $S_{1,\overline{1}} := \mathrm{X}_{1,\overline{1}} + \mathrm{X}_{2,\overline{2}}$.

Homogeneous coordinate ring. Let it be denoted by $\mathbb{C}[\mathrm{Sp}\mathcal{F}_{2n}]$. Then:

$$\mathbb{C}[\mathrm{X_J}]/\mathrm{I} = \mathbb{C}[\mathrm{Sp}\mathcal{F}_{2n}] = \bigoplus_{\lambda \in P^+} \mathbb{C}[\mathrm{Sp}\mathcal{F}_{2n}]_\lambda \simeq \bigoplus_{\lambda \in P^+} \mathrm{V}_\lambda^*.$$

The direct sum of the dual modules is an algebra because of the existence of the embedding of modules: $\mathrm{V}_{\lambda+\mu} \hookrightarrow \mathrm{V}_\lambda \otimes \mathrm{V}_\mu$, $\nu_{\lambda+\mu} \mapsto \nu_\lambda \otimes \nu_\mu$.

The PBW Degenerate Symplectic Grassmann Variety.
Let $\mathrm{W} := \mathbb{C}^{2n}$ and let:

$$\mathrm{W} = \mathrm{W}_{k,1} \oplus \mathrm{W}_{k,2} \oplus \mathrm{W}_{k,3},$$

where:
$\mathrm{W}_{k,1} = \mathrm{span}(\mathrm{w}_1, \ldots, \mathrm{w}_k)$,
$\mathrm{W}_{k,2} = \mathrm{span}(\mathrm{w}_{k+1}, \ldots, \mathrm{w}_{2n-k})$,
$\mathrm{W}_{k,3} = \mathrm{span}(\mathrm{w}_{2n-k+1}, \ldots, \mathrm{w}_{2n})$.

Let $\mathrm{pr}_{1,3}$ denote the projection $\mathrm{pr}_{1,3} : \mathrm{W} \to \mathrm{W}_{k,1} \oplus \mathrm{W}_{k,3}$, i.e.,

$$\mathrm{pr}_{1,3}(w_1, \dots, w_{2n}) = (w_1, \dots, w_k, 0, \dots, 0, w_{2n-k+1}, \dots, w_{2n}).$$

Proposition (Feigin, Finkelberg and Littelmann '13)

The PBW degen. sympl. Grassmann variety $\mathrm{SpGr}^a(k, 2n)$ is given by:
$\mathrm{SpGr}^a(k, 2n) = \{\mathrm{U} \in \mathrm{Gr}(k, 2n) \mid \mathrm{pr}_{1,3}(\mathrm{U}) \text{ is isotropic}\}$.

Remark. We don't have $\mathrm{SpGr}(k, 2n) \simeq \mathrm{SpGr}^a(k, 2n)$ any more.
Exception: case for $k = n$.

The PBW degenerate complete symplectic flag variety. For $\lambda \in P^+$, let it be denoted by $\mathrm{Sp}\mathcal{F}^a_{2n}$. We have $\mathrm{Sp}\mathcal{F}^a_{2n} = \overline{\mathrm{Sp}^a_{2n}[\nu^a_\lambda]} \hookrightarrow \mathbb{P}(\mathrm{V}^a_\lambda)$.

On the other hand, denote by $\mathrm{pr}_k : \mathrm{W} \to \mathrm{W}$ the projections along w_k, i.e.,

$$\mathrm{pr}_k\Big(\sum_{j=1}^{2n} c_j \mathrm{w}_j\Big) = \sum_{j \neq k} c_j \mathrm{w}_j.$$

Theorem (Feigin, Finkelberg and Littelmann '13)

$\mathrm{Sp}\mathcal{F}^a_{2n}$ is naturally embedded in $\prod_{k=1}^{n} \mathrm{SpGr}^a(k, 2n)$. The image is equal to the subvariety formed by the collections $(\mathrm{U}_k)_{k=1}^{n}$ such that $pr_{k+1}\mathrm{U}_k \subset \mathrm{U}_{k+1}, \; k = 1, \ldots, n-1$ and $\mathrm{U}_k \in \mathrm{SpGr}^a(k, 2n)$.

It is a flat degeneration of $\mathrm{Sp}\mathcal{F}_{2n}$ and it is an irreducible variety.

Degenerate relations. Let $\mathbb{C}[\mathrm{X}^a_{\mathrm{J}}]$ be the polynomial ring in variables $\mathrm{X}^a_{\mathrm{J}}$ for all $\mathrm{J} = (j_1 < \cdots < j_k) \subset \{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$, $k = 1, \ldots, n$.

Definition

The PBW degree of J (and hence of X_{J}) is given by

$$\deg \mathrm{J} = \#\{r \mid j_r > k\}.$$

- Degenerate Plücker relations (Feigin. '12). These are obtained by picking out the terms of minimal PBW degree from $R^t_{\mathrm{L},\mathrm{J}}$ and applying the map $\mathrm{X}_{\mathrm{J}} \mapsto \mathrm{X}^a_{\mathrm{J}}$. They are given by

$$R^{t;a}_{\mathrm{L},\mathrm{J}} := \mathrm{X}^a_{\mathrm{L}}\mathrm{X}^a_{\mathrm{J}} - \sum \mathrm{X}^a_{\mathrm{L}'}\mathrm{X}^a_{\mathrm{J}'}, \tag{1.2}$$

labeled by the sequences L, J; $n \geq |\mathrm{L}| \geq |\mathrm{J}| \geq 1$ and a number t; $1 \leq t \leq |\mathrm{J}|$.

- Degenerate symplectic linear relations. Let $\mathrm{U}_2 \subset \mathbb{C}^6$, then: $\mathrm{pr}_{1,3}(u) = a_{1,1}\mathrm{w}_1 + a_{2,1}\mathrm{w}_2 + a_{5,1}\mathrm{w}_5 + a_{6,1}\mathrm{w}_6$ and $\mathrm{pr}_{1,3}(v) = a_{1,2}\mathrm{w}_1 + a_{2,2}\mathrm{w}_2 + a_{5,2}\mathrm{w}_5 + a_{6,2}\mathrm{w}_6$.

  Then $\mathrm{pr}_{1,3}(\mathrm{U}_2)$ is isotropic if and only if $\mathrm{pr}_{1,3}(u)^T \mathrm{M}_s \mathrm{pr}_{1,3}(v) = 0$, i.e.,

$$-a_{6,1}a_{1,2} - a_{5,1}a_{2,2} + a_{2,1}a_{5,2} + a_{1,1}a_{6,2} = 0,$$

  which leads to the degenerate symplectic linear relation $\mathrm{X}^a_{1,6} + \mathrm{X}^a_{2,5} = 0$ (or $\mathrm{X}^a_{1,\overline{1}} + \mathrm{X}^a_{2,\overline{2}} = 0$).

  Ones gets the same relation by picking out terms of minimal PBW degree from $\mathrm{X}_{1,\overline{1}} = -\mathrm{X}_{2,\overline{2}} - \mathrm{X}_{3,\overline{3}}$. Denote these relations by S^a_{L}.

Main result. Let I^a be the ideal in $\mathbb{C}[\mathrm{X}^a_{\mathrm{J}}]$ generated by $R^{t;a}_{\mathrm{L},\mathrm{J}}$ and S^a_{L}.

Theorem (B. '20)

The ideal I^a *is the defining ideal of* $\mathrm{Sp}\mathcal{F}^a_{2n}$.

Example

For $\mathrm{Sp}\mathcal{F}^a_4$, the ideal I^a is generated by the relations:

$R^{1;a}_{(1,2),(\overline{2})} = \mathrm{X}^a_{1,2}\mathrm{X}^a_{\overline{2}} + \mathrm{X}^a_{2,\overline{2}}\mathrm{X}^a_{1}$,

$R^{1;a}_{(1,2),(\overline{1})} = \mathrm{X}^a_{1,2}\mathrm{X}^a_{\overline{1}} + \mathrm{X}^a_{2,\overline{1}}\mathrm{X}^a_{1}$,

$R^{1;a}_{(1,\overline{2}),(\overline{1})} = \mathrm{X}^a_{1,\overline{2}}\mathrm{X}^a_{\overline{1}} + \mathrm{X}^a_{\overline{2},\overline{1}}\mathrm{X}^a_{1} - \mathrm{X}^a_{1,\overline{1}}\mathrm{X}^a_{\overline{2}}$,

$R^{1;a}_{(1,2),(\overline{2},\overline{1})} = \mathrm{X}^a_{1,2}\mathrm{X}^a_{\overline{2},\overline{1}} - \mathrm{X}^a_{1,\overline{2}}\mathrm{X}^a_{2,\overline{1}} + \mathrm{X}^a_{1,\overline{1}}\mathrm{X}^a_{2,\overline{2}}$,

$R^{1;a}_{(2,\overline{2}),(\overline{1})} = \mathrm{X}^a_{2,\overline{2}}\mathrm{X}^a_{\overline{1}} - \mathrm{X}^a_{2,\overline{1}}\mathrm{X}^a_{\overline{2}}$,

and the linear relation $S^a_{(1,\overline{1})} = \mathrm{X}^a_{1,\overline{1}} + \mathrm{X}^a_{2,\overline{2}}$.

Homogeneous coordinate ring. Let it be denoted by $\mathbb{C}[\mathrm{Sp}\mathcal{F}^a_{2n}]$. Let J^a be the actual defining ideal of $\mathrm{Sp}\mathcal{F}^a_{2n}$. Then we have the equality, direct sum decompositions and isomorphism

$$\mathbb{C}[\mathrm{X}^a_{\mathrm{J}}]/\mathrm{J}^a = \mathbb{C}[\mathrm{Sp}\mathcal{F}^a_{2n}] = \bigoplus_{\lambda \in P^+} \mathbb{C}[\mathrm{Sp}\mathcal{F}^a_{2n}]_\lambda \simeq \bigoplus_{\lambda \in P^+} (\mathrm{V}^a_\lambda)^*.$$

- The algebra structure on the direct sum of the dual modules is due to existence of a unique injective homomorphism of modules, $\mathrm{V}^a_{\lambda+\mu} \hookrightarrow \mathrm{V}^a_\lambda \otimes \mathrm{V}^a_\mu, \quad \nu^a_{\lambda+\mu} \mapsto \nu^a_\lambda \otimes \nu^a_\mu$ (Feigin, Fourier and Littelmann, 2011).
- The isomorphism $\mathbb{C}[\mathrm{Sp}\mathcal{F}^a_{2n}]_\lambda \simeq (\mathrm{V}^a_\lambda)^*$ is due to Feigin, Finkelberg and Littelmann, 2013.

2. The symplectic semistandard PBW tableaux.

To a dominant weight $\lambda = \sum_{k=1}^{n} m_k \omega_k$, associate the partition $\lambda = (m_1 + m_2 + \cdots + m_n, m_2 + \cdots + m_n, \ldots, m_n)$.

Question:
What is the set of tableaux that labels a basis for both $\mathbb{C}[\mathrm{Sp}\mathcal{F}_{2n}]$ and $\mathbb{C}[\mathrm{Sp}\mathcal{F}^a_{2n}]$?

Young tableaux:

Definition (Young '28)

1	3	3
3	$\overline{2}$	
$\overline{1}$		

A semistandard Young tableau of shape $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ on $\mathcal{N} := \{1, \ldots, n, \overline{n}, \ldots, \overline{1}\}$ is a filling of the corresponding Young diagram with $\mathrm{T}_{i,j} \in \mathcal{N}$ such that:

$$i_1 < i_2 \Rightarrow \mathrm{T}_{i_1,j} < T_{i_2,j} \quad \text{and} \quad j_1 < j_2 \Rightarrow \mathrm{T}_{i,j_1} \leq \mathrm{T}_{i,j_2}.$$

Do these tableaux fit the role?

Definition (Feigin '12)

1
$\bar{1}$
3

A PBW tableau is a filling $\mathrm{T}_{i,j}$ such that:

if $\mathrm{T}_{i,j} \leq \mu_j$, then $\mathrm{T}_{i,j} = i$, and

if $i_1 < i_2$ and $\mathrm{T}_{i_1,j} \neq i_1$, then $\mathrm{T}_{i_1,j} > \mathrm{T}_{i_2,j}$.

1	$\bar{2}$	3
$\bar{1}$	3	
3		

It is semistandard if in addition:

for any $j > 1$ and any i, $\exists\ i_1 \geq i$ such that $\mathrm{T}_{i_1,j-1} \geq \mathrm{T}_{i,j}$.

Definition (B. '20)

$\bar{2}$
2
3

A semistandard PBW tableau is symplectic if:

whenever $\exists\, i, i_1$ such that $\mathrm{T}_{i,j} = i$ and $\mathrm{T}_{i_1,j} = \bar{i}$, then $i_1 < i$.

Example

For $\mathfrak{g} = \mathfrak{sp}_4$, and dominant weight $\lambda = \omega_1 + \omega_2$, we have:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \overline{2} \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \overline{2} & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \overline{2} & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \overline{2} & \overline{2} \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},$$

$$\begin{array}{|c|c|}\hline \overline{1} & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \overline{1} & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \overline{1} & \overline{2} \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \overline{1} & \overline{1} \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \overline{1} & 1 \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \overline{1} & 2 \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \overline{1} & \overline{2} \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \overline{1} & \overline{1} \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Symplectic linear relations, again!

For L non symplectic, X_{L} can be written as a linear combination of $\mathrm{X}_{\mathrm{L}'}$ with L' symplectic, i.e., we have the relations: $S_{\mathrm{L}} := \mathrm{X}_{\mathrm{L}} - \sum \mathrm{X}_{\mathrm{L}'}$.
Recall again $S_{(1,\overline{1})} := \mathrm{X}_{1,\overline{1}} + \mathrm{X}_{2,\overline{2}} + \mathrm{X}_{3,\overline{3}}$.

Lemma (B. '20)

The PBW degree of the element $\mathrm{X_L}$ in S_L with L non symplectic, is less or equal to the PBW degrees of the other terms.

Symplectic semistandard PBW tableaux - FFLV basis bijection.
For $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $\lambda \in P^+$, recall the $\mathfrak{sp}_{2n}$ and $\mathfrak{sp}_{2n}^a$-modules V_λ and V_λ^a respectively.

Let R^+ be the set of positive roots of $\mathfrak{sp}_{2n}$. For each $\alpha \in R^+$, fix a non zero element $f_\alpha \in \mathfrak{n}^-_{-\alpha}$. All positive roots of $\mathfrak{sp}_{2n}$ can be written as:

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j, \qquad 1 \le i \le j \le n,$$
$$\alpha_{i,\bar{j}} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_n + \alpha_{n-1} + \ldots + \alpha_j, \quad 1 \le i \le n < j, \; i + j \le 2n,$$

where $\alpha_{i,n} = \alpha_{i,\bar{n}}$. We write $f_{i,j}$ instead of $f_{\alpha_{i,j}}$. Notice also that $\alpha_i := \alpha_{i,i}$ and $\alpha_{\bar{j}} := \alpha_{j,\bar{j}}$.

Definition (Feigin, Fourier and Littelmann '11)

A symplectic Dyck path is a sequence $\mathrm{p} = (p(0), \dots, p(k))$, $k \geq 0$, of positive roots such that:

(i) $p(0)$ is simple and $p(k) = \alpha_j$ or $p(k) = \alpha_{\bar{j}}$ and

(ii) if $p(s) = \alpha_{p,q}$, then $p(s+1) = \alpha_{p,q+1}$ or $p(s+1) = \alpha_{p+1,q}$.

Denote by $\mathbb{D}$ the set of all Dyck paths. For $\lambda = \sum_{i=1}^{n} m_i \omega_i \in P^+$, the symplectic Feigin-Fourier-Littelmann-Vinberg (FFLV) polytope $\mathrm{P}(\lambda) \subset \mathbb{R}_{\geq 0}^{n^2}$ is the polytope $\mathrm{P}(\lambda) := \{(s_\alpha)_{\alpha > 0}, \forall p \in \mathbb{D}\}$, such that:

$$\begin{cases} s_{p(0)} + \ldots + s_{p(k)} \leq m_i + \ldots + m_j, & p(0) = \alpha_i, \quad p(k) = \alpha_j, \\ s_{p(0)} + \ldots + s_{p(k)} \leq m_i + \ldots + m_n, & p(0) = \alpha_i, \quad p(k) = \alpha_{\bar{j}}, \\ s_{p(i)} \geq 0, & 0 \leq i \leq k. \end{cases}$$

Let $\mathrm{S}(\lambda)$ be the set of integral points in $\mathrm{P}(\lambda)$. For a multi-exponent $s = (s_\beta)_{\beta>0}$, $s_\beta \in \mathbb{Z}_{\geq 0}$, let f^s be the element: $f^s = \prod_{\beta \in \mathbb{R}^+} f_\beta^{s_\beta} \in \mathcal{S}(\mathfrak{n}^-)$.

Theorem (Feigin, Fourier and Littelmann '11)

The elements $\{f^s \nu_\lambda^a, s \in \mathrm{S}(\lambda)\}$ *form a basis of* V_λ^a *and hence of* V_λ.

We call $\pi_\lambda := \{f^s \nu_\lambda, s \in \mathrm{S}(\lambda)\}$ the *symplectic FFLV basis*.

Example

For $\mathfrak{sp}_4$ and $\lambda = \omega_1 + \omega_2$, the 16 integral points of $\mathrm{P}(\lambda) \subset \mathbb{R}^4$ give rise to the set of monomials:
$\{1, f_{11}, f_{22}, f_{11}f_{22}, f_{12}f_{22}, f_{12}, f_{11}f_{12}, f_{12}^2, f_{1\bar{1}}, f_{11}f_{1\bar{1}}, f_{12}f_{1\bar{1}}, f_{1\bar{1}}^2, f_{1\bar{1}}f_{2\bar{2}}, f_{11}f_{1\bar{1}}f_{22},$
$f_{12}f_{1\bar{1}}f_{22}, f_{1\bar{1}}^2 f_{22}\}$.

Proposition (B. '20)

The symplectic FFLV basis π_λ for V_λ and V_λ^a is in bijection with the set SyST_λ of symplectic semistandard PBW tableaux of shape λ.

Illustrative example:

(Say $f_{i_1,j_1} > f_{i_2,j_2}$ if either $i_1 < i_2$ or $i_1 = i_2$ and $j_1 > j_2$).

For $\mathfrak{sp}_4$, consider the "action" of the ordered product $f_{12}f_{1\bar{1}}f_{22}$ on the highest weight tableau of shape $\lambda = (2,1)$ as seen below:

$$f_{12}f_{1\bar{1}}f_{22}\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}\right) = f_{12}f_{1\bar{1}}\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \bar{2} & \\ \cline{1-1}\end{array}\right) = f_{12}\left(\begin{array}{|c|c|}\hline \bar{1} & 1 \\ \hline \bar{2} & \\ \cline{1-1}\end{array}\right) = \begin{array}{|c|c|}\hline \bar{1} & \bar{2} \\ \hline \bar{2} & \\ \cline{1-1}\end{array}.$$

3. Proof of the main result

To each $\mathrm{T} \in \mathsf{SyST}_\lambda$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$, associate respectively the elements $\mathrm{X}_\mathrm{T} \in \mathrm{V}_\lambda^*$ and $\mathrm{X}_\mathrm{T}^a \in (\mathrm{V}_\lambda^a)^*$ via:

$$\mathrm{T} \mapsto \mathrm{X}_\mathrm{T} = \prod_{j=1}^{\lambda_1} \mathrm{X}_{\mathrm{T}_{1,j},\ldots,\mathrm{T}_{\mu_j,j}} \quad \text{and} \quad \mathrm{T} \mapsto \mathrm{X}_\mathrm{T}^a = \prod_{j=1}^{\lambda_1} \mathrm{X}^a_{\mathrm{T}_{1,j},\ldots,\mathrm{T}_{\mu_j,j}}.$$

Theorem (B. '20)

The set of tableaux $\{\mathrm{T} \mid \mathrm{T} \in \mathit{SyST}_\lambda\}$ labels a basis for $\mathbb{C}[\mathrm{S}\mathrm{P}\mathcal{F}_{2n}^a]_\lambda$ and $\mathbb{C}[\mathrm{S}\mathrm{P}\mathcal{F}_{2n}]_\lambda$.

Idea of the proof: We have $\#\{\mathrm{T} \mid \mathrm{T} \in \mathsf{SyST}_\lambda\} = \dim \mathrm{V}_\lambda = \dim \mathrm{V}_\lambda^a$.

We consider only $\mathbb{C}[\mathrm{S}\mathrm{P}\mathcal{F}_{2n}^a]_\lambda$. It suffices to show that the elements X_T^a span $\mathbb{C}[\mathrm{S}\mathrm{P}\mathcal{F}_{2n}^a]_\lambda$.

- Start with $\mathrm{X}^a_{\mathrm{T}}$, T symplectic but not semistandard. Consider two arbitrary columns L, J of T that violate this condition.

- The term $\mathrm{X}^a_{\mathrm{L}}\mathrm{X}^a_{\mathrm{J}}$ remains in $R^{t;a}_{\mathrm{L},\mathrm{J}}$.

- Applying $R^{t;a}_{\mathrm{L},\mathrm{J}}$ leads to smaller tableaux w.r.t a fixed total order.

- Now apply relations S^a_{L} to replace non symplectic columns. The terms in the summands become smaller w.r.t the above fixed order.

Theorem (B. '20)

The ideal I^a defined before is the defining ideal of $\mathrm{Sp}\mathcal{F}^a_{2n}$ and it is prime.

Thank you very much for your attention!

Some references

- G. Cerulli Irelli, X. Fang, E. Feigin, G. Fourier and M. Reineke. *Linear degenerations of flag varieties: partial flags, defining equations, and group actions.* Mathematatische Zeitschrift (2019): 1-25 [DOI10.1007/s00209-016-1839-y].
- R. Chirivi, P. Littelmann and A. Maffei. *Equations defining symmetric varieties and affine Grassmannians.* International Mathematics Research Notices 2009, no. 2 (2009): 291-347.
- C. De Concini. *Symplectic standard tableaux*. Advances in Mathematics, 34(1) (1979), 1-27.
- E. Feigin. $\mathbb{G}_M^a$ *Degenerations of flag varieties*, Selecta Mathematica 18, no. 3 (2012): 513-537

E. Feigin, M. Finkelberg and P. Littelmann. *Symplectic degenerate flag varieties*. Canadian Journal of Mathematics, 66(6) (2014), 1250-1286.

E. Feigin, G. Fourier, and P. Littelmann. *PBW Filtration and Bases for symplectic Lie Algebras*, Internat. Math. Res. Notices (2011), Vol. 2011, 5760–5784.