steps arrive to $l' \in \mathcal{F}_\omega L \setminus \mathcal{F}_{\omega *} L$. For $m' = m - \varphi(l)$ we have $\psi(m') = \psi(m) = f$. By construction: $m' \in \mathcal{F}_{\omega'} M$ with $m' = 0$ or $\omega' < \omega$.
We can proceed by $\omega > \omega' > \ldots > \gamma$.

$\tilde{m} - m = \varphi(l) + \varphi(l') + \ldots \in \varphi(L) \subset \ker(\psi)$.

Since $\psi(m) \neq 0 \Rightarrow \tilde{m} \notin \varphi(L)$. At each step with $m^{(i)} \neq 0$ we terminate at exactly $\gamma$. Hence $f = \psi(m) = \psi(\tilde{m})$. $\square$

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<u>proof of proposition 7(ii):</u> $\boxed{15.05.2019}$

$$(\mathrm{gr}^{\tilde{\mathcal{F}}} M)_\gamma = \sum_{\delta_i + \gamma_i \leq \gamma} (\mathrm{gr}^{\mathcal{F}} A)_{\delta_i} \cdot \sigma(\xi_i), \text{ then } \forall m \in \tilde{\mathcal{F}}_\gamma M : m = \sum_{\delta_i + \gamma_i \leq \gamma} a_{\delta_i} \cdot \xi_i + m',$$
$$m' \in \tilde{\mathcal{F}}_{\gamma *} M, \quad a_{\delta_i} \in \mathcal{F}_{\delta_i} A.$$

Suppose $m' \neq 0$ and $\deg(m') = \gamma' < \gamma$. Then $m' = \sum a_{\varepsilon_i} \cdot \xi_i + m''$ with $m'' = 0$ or $m'' \in \mathcal{F}_{\gamma' *} M$, $\gamma > \gamma' > \gamma'' \ldots$

We stop after finitely many steps ($\Gamma$ is well-ordered) $\Rightarrow$ some $m^\circ = 0$.

$$\Rightarrow m \in \sum_{j \in J} \sum_{\delta_j + \gamma_j \leq \gamma} (\mathcal{F}_{\delta_j} A) \cdot \xi_j = \tilde{\mathcal{F}}_\gamma M. \Rightarrow M = \sum_{j \in J} A \cdot \xi_j. \quad \square$$

<u>proof of theorem 6</u>

<u>(i) $\Rightarrow$ (ii):</u> $(\mathrm{gr}\,*)$ is a complex, i.e. $\mathrm{Im}(\mathrm{gr}\,\varphi) \subseteq \ker(\mathrm{gr}\,\psi)$. We need to prove "$\supseteq$". Let $m \in \mathcal{F}_\gamma M \setminus \mathcal{F}_{\gamma *} M$, suppose $m \in \ker(\mathrm{gr}\,\psi) \Leftrightarrow$
$0 = [\psi(m)] = (\mathrm{gr}\,\psi)[m]$.

If $\psi(m) = 0 \Rightarrow m \in \ker \psi = \mathrm{Im}\,\varphi \Rightarrow \exists\, l \in L : m = \varphi(l) \in \varphi(L) \cap \mathcal{F}_\gamma M = \varphi(\mathcal{F}_\gamma L)$
[by strictness of $\varphi$]

If $\psi(m) \neq 0$, since $[\psi(m)] = 0$ $\exists\, \gamma' < \gamma : \psi(m) \in \mathcal{F}_{\gamma'} N$.

$\psi(m) \in \psi(M) \cap \mathcal{F}_{\gamma'} N \Rightarrow \exists\, m' \in \mathcal{F}_{\gamma'} M : \psi(m) = \psi(m') \Rightarrow m - m' \in \ker(\psi) \cap \mathcal{F}_\gamma M$
$= \mathrm{Im}(\varphi) \cap \mathcal{F}_\gamma M$
$\overset{(*)\text{exact}}{=} \varphi(L) \cap \mathcal{F}_\gamma M = \varphi(\mathcal{F}_\gamma L)$

$\Rightarrow m = m' + \varphi(l')$, $l' \in \mathcal{F}_\gamma L$, then
$[m] = [m - m'] = [\varphi(l')] = (\mathrm{gr}\,\varphi)[l'] \Rightarrow [m] \in \mathrm{Im}(\mathrm{gr}\,\varphi)$ $\square$

## META THEOREM 8: "Filtered graded transfer"

Let $(*, A, M)$ denotes some property $*$ of an A-Module $M$.
If $M$ is a $\Gamma$-filtered A-module for $A$ a $\Gamma$-filtered ring with compatible filtrations $\mathcal{F}M$, $\mathcal{F}A$, then

" $(*, gr^{\mathcal{F}}A, gr^{\mathcal{F}}M)$ implies $(*, A, M)$ "

It is valid, in particular, for the following properties:

a) left / right Noetheriarrity
b) left / right Artiniarity
c) simplicity of an algebra [algebra A simple $\Leftrightarrow$ the only two-sided ideals are $0$ and $\langle 1 \rangle$]
d) simplicity of a module [module M is simple $\Leftrightarrow$ the only submodules are $0$ and $M$]
e) semisimplicity of an Artinian algebra ($\Leftrightarrow J(A) = 0$)
$\qquad J(A) = \bigcap\{m \mid m$ is left/right maximal ideal$\}$
f) primeness / semiprimeness / domain prop's.

ring $R$: prime: $\forall a,b \in R$ : $aRb = \{arb \mid r \in R\} = 0 \Rightarrow a = 0 \vee b = 0$
$\qquad$ [$\forall$ ideals $A, B \subseteq R$ : $AB = 0 \Rightarrow A = 0 \vee B = 0$]
semiprime: $\forall x \in R \setminus 0$, $xRx \neq \{0\}$
$\qquad$ [$\forall$ ideal $A \subseteq R$ $\exists k \in \mathbb{N}$ : $A^k = 0 \Rightarrow A = 0$ .]

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**Proposition 9:** (i) Let $\varphi: M \longrightarrow N$ a $\Gamma$-filtered homomorphism of $\Gamma$-filtered left [right] modules $M, N$. Then

$gr^{\mathcal{F}}\varphi$ is injective (surjective) $\Leftrightarrow$ $\varphi$ is injective (surjective)

(ii) Suppose that $L, N \subseteq M$ be $\Gamma$-filterated w.r.t $\mathcal{F}M$.
Consider the $\Gamma$-filtrations $\mathcal{F}N = \{\mathcal{F}_\gamma M \cap N\}_\gamma$, $\mathcal{F}L = \{\mathcal{F}_\gamma M \cap L\}_\gamma$.
If $L \subseteq N$, then $gr^{\mathcal{F}}L \subseteq gr^{\mathcal{F}}N$. Moreover, if $gr^{\mathcal{F}}L = gr^{\mathcal{F}}N \Rightarrow L = N$.

*proof:* (i) $0 \to \operatorname{Ker}\varphi \to M \longrightarrow N \to \operatorname{coker}\varphi \to 0$ , apply thm 6:

$gr\,\varphi$ is injective $\Leftrightarrow \operatorname{Ker}(gr\,\varphi) = 0 \Leftrightarrow 0 \to grM \xrightarrow{gr\varphi} grN \to \operatorname{coker}(gr\varphi) \to 0$ is exact

$\Leftrightarrow 0 \to M \xrightarrow{\varphi} N \to L \to 0$ is exact with $\varphi$ strict. and $L$ s.t. $grL = \operatorname{coker}(gr\,\varphi)$

$\Leftrightarrow \varphi$ is injective and strict.

(ii) $L \subseteq N \Rightarrow 0 \to L \xrightarrow{i} N$ is exact. Since $\mathcal{F}L, \mathcal{F}N$ are induced from $\mathcal{F}M$, $i$ is strict.
By (i) $0 \to grL \xrightarrow{gr\,i} grN$ is exact $\Rightarrow grL \subseteq grN$ and
if $grL = grN \Rightarrow gr\,i$ is surjective, i.e. bijective $\Rightarrow i$ is bijective
$\Rightarrow L = N$. $\square$

$\boxed{\text{supplement:}}$ The finest filtration on $K[x_1,\dots,x_n]=R$

$x_1^{\alpha_1}\cdots x_n^{\alpha_n} =: x^{\alpha}$, $\alpha\in\mathbb{N}_0^n$. $\mathbb{N}_0^n \longleftrightarrow \{x^\alpha \mid \alpha\in\mathbb{N}_0^n\}$ is a K-basis of R.

Definition: Let $<$ be a total ordering on R.

It is a monomial ordering, if $x^\alpha < x^\beta :\Leftrightarrow \alpha<\beta$ on $(\mathbb{N}_0^n,0,+)$

and $\forall \gamma\in\mathbb{N}_0^n: \alpha+\gamma<\beta+\gamma$. $(\Rightarrow x^\gamma\cdot x^\alpha < x^\gamma\cdot x^\beta)$

It is a well ordering, if any subset of $(\mathbb{N}_0^n,0,+)$ has a min. element wrt $<$.

Examples: a) lexicographical ord.: $\alpha <_{lex} \beta \Leftrightarrow \exists k: \alpha_1=\beta_1,\dots,\alpha_k=\beta_k, \alpha_{k+1}<\beta_{k+1}$;

$\rightsquigarrow x>y>z$, $x>y\cdot z$, $x>y^{2019}$

b) weighted degree: for $w_i\in R$, $w=(w_1,\dots,w_n)$: $\deg_w x^\alpha=\sum w_i\alpha_i$

$\rightsquigarrow w=(1,\dots,1) \Rightarrow \deg_w=\deg$

$w=(2,3) \Rightarrow x^3-y^2$ is graded in w-degree 6

extension of some ordering $<$ is $(<_w)$:

$x^\alpha <_w x^\beta \Leftrightarrow \deg_w x^\alpha<\deg_w x^\beta$ or $\deg_w x^\alpha=\deg_w x^\beta$ and $x^\alpha<x^\beta$.

c) reverse lex.: $\alpha <_{rlex}\beta \Leftrightarrow \exists k: \alpha_n=\beta_n,\dots,\alpha_k=\beta_k, \alpha_{k-1}<\beta_{k-1}$

popular orderings: degree-lex, degree-revlex

Remark: $<$ is global $\Rightarrow 1=x^{\underline{0}}$ is the smallest element w.r.t. $<$ on $\mathbb{N}_0^n$.

$<_w$ is global $\Leftrightarrow w_i\geq 0$, usually $w_i\in\mathbb{N}_0$

Let $w=(w_1,\dots,w_n)\in\mathbb{N}^n$ be given, define $\mathcal{F}_\gamma R=\{\sum Kx^\alpha \mid \alpha\leq_w\gamma\}$, $\gamma\in\mathbb{N}_0^n$

$<_w$ a w-extension of some $<$

$(\mathcal{F}_\alpha R)(\mathcal{F}_\beta R)\subseteq \mathcal{F}_{\alpha+\beta}R$

$\tilde{\mathcal{F}}_d=\{f\in K[x_1,\dots,x_n] \mid \deg_w f\leq d\}$, $d\in\mathbb{N}_0$

$(gr^{\mathcal{F}}R)_\beta=K[x^\beta]$

$K[x,y]$, $w=(2,3)$, $x>y$ lex and $x^\alpha<x^\beta$

$$\tilde{\mathcal{F}}_6=K\cdot x^3+K\cdot y^2+\tilde{\mathcal{F}}_{6^*}=\mathcal{F}_{(3,0)} \supseteq \mathcal{F}_{(2,0)}+\mathcal{F}_{(0,0)^*}$$

$$\mathcal{F}_{(3,0)} = Kx^3+\mathcal{F}_{(3,0)^*}$$