Classical Ansatz:

we use coefficients at monomials as variables

$x^2+3x+42=(x-a)(x-b)=x^2-(a+b)x+ab \quad \Rightarrow \quad \begin{array}{l} a+b=-3 \\ a\cdot b = 42 \end{array}$

Definition ⓐ Let A be a Γ-graded algebra and M be a left A-module ($\forall m\in M, a\in A: a\cdot m \in M$). M is called a Γ-graded left A-module, if

(i) $M=\bigoplus_{\gamma\in\Gamma} M_\gamma$

(ii) $A_\alpha M_\gamma \subseteq M_{\alpha\circ\gamma} \quad \forall \alpha,\gamma\in\Gamma$

ⓑ Let A,B be two Γ-graded K-algebras, $\varphi: A=\bigoplus_{\gamma\in\Gamma} A_\gamma \longrightarrow B=\bigoplus_{\gamma\in\Gamma} B_\gamma$ a homomorphism of K-algebras. φ is called a Γ-graded homomorphism, if $\forall \gamma\in\Gamma: \varphi(A_\gamma)\subseteq B_\gamma$, [since: $\varphi(A_\alpha A_\gamma)=\varphi(A_\alpha)\cdot\varphi(A_\gamma)\subseteq \varphi(A_{\alpha\circ\gamma})$, $\varphi(A_0)\subseteq B_0$]

ⓒ Let M,N be two Γ-graded A-modules, $\psi: M=\bigoplus M_\gamma \longrightarrow N=\bigoplus N_\gamma$ a homomorphism of A-modules. ψ is called a Γ-graded homomorphism with shift $\delta\in\Gamma$, if $\psi(M_\gamma)\subseteq N_{\gamma\circ\delta}$ $\forall\gamma\in\Gamma$.

Example: $I(f(x)) = \int_0^x f(t)\,dt$

algebra generated by x,∂,I : $\begin{cases} I\cdot x = x\cdot I - I^2 \\ \partial\cdot x = x\partial + 1 \\ \partial\cdot I = 1 \end{cases}$

$(I\partial)(f(x)) = \int_0^x \frac{\partial f}{\partial t}(t)\,dt = f(x)-f(0)$

$(1-I\partial)(f) = f(0)$ Evaluation

$(1-I\partial)^2 = (1-I\partial)$

Homework 2: $\mathbb{Z}$-grading?

Solution: (Homework) 24.04.2019

1.) $\operatorname{im}(\tau) = \sum_{i,j\in\mathbb{N}_0} K\cdot x^i\partial^{i+j} = \sum_{i,j\in\mathbb{N}_0} K\cdot(x\partial)^i\partial^j = K\langle x\partial,\partial \mid \partial\cdot x\partial = x\partial^2+\partial\rangle$

$D = \underbrace{\bigoplus_{j=0}^{\infty} K[x\partial]x^j}_{\text{"negative part"}} \oplus \underbrace{K[x\partial]}_{\deg 0} \oplus \underbrace{\bigoplus_{i=0}^{\infty} K[x\partial]\partial^i}_{\text{"positive part"}}$

$\ker(\tau) \ni p = \sum c_{ij} k^i s_k^j$; $\tau(p) = \sum c_{ij}(x\partial)^i\partial^j = \sum_j \underbrace{\left(\sum_i c_{ij}(x\partial)^i\right)}_{=: q_j(x\partial)}\partial^j$

$\tau(p)=0 \iff q_j(x\partial)=0 \quad \forall j$

holds also in $K[\theta]$.

2.) $A = K\langle x, \partial, I \mid \underbrace{Ix = xI - I^2}_{(i)}, \underbrace{\partial x = x\partial + 1}_{(ii)}, \underbrace{\partial I = 1}_{(iii)} \rangle$

$\{x^i I^j \partial^k\}$

$$\left.\begin{array}{l} \deg x = a \\ \deg \partial = b \\ \deg I = c \end{array}\right\} \qquad \begin{array}{l} (i) \Rightarrow a + c = 2c \Rightarrow a = c \\ (ii) \Rightarrow a = -b \\ (iii) \Rightarrow b = -c \end{array}$$

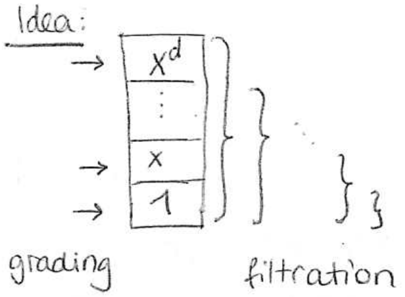
all $\mathbb{Z}$-gradings of A are $\mathbb{Z}\cdot(1,-1,1)$.

$A_0 \supseteq K[x\partial]$, $I\partial$, $\partial I \equiv 1$, $(I\partial)^2 = (I\cdot\partial)(I\cdot\partial) = I\cdot\partial$

$xI\partial^2, \dots, x^i I^j \partial^{i+j}$; $I^i x^j \partial^k$

Filtered algebras and modules

Idea:



Let K be a field, $\Gamma = (\Gamma, \circ, e)$ a monoid, totally ordered by $<$, which is compatible with $\circ$: $\alpha < \beta \Rightarrow \alpha\circ\gamma < \beta\circ\gamma \quad \forall \gamma \in \Gamma$.

Definition 1: A K-algebra A is Γ-filtered if $\exists$ K-vectorspaces $A_\gamma = \mathcal{F}_\gamma A = \mathcal{F}_\gamma$ ← different notations such that

(i) $A = \sum_{\gamma\in\Gamma} \mathcal{F}_\gamma A$ "exhaustiveness"

(ii) $\forall \alpha < \beta$ one has $\mathcal{F}_\alpha A \subseteq \mathcal{F}_\beta A$ "ascending property"

(iii) $\forall \alpha < \beta$ $(\mathcal{F}_\alpha A)\cdot(\mathcal{F}_\beta A) \subseteq \mathcal{F}_{\alpha\circ\beta} A$.

By assumption, $K \subseteq \mathcal{F}_e A$; $\mathcal{F} = \{\mathcal{F}_\gamma \mid \gamma \in \Gamma\}$

Note: For $B \subset A$ subalgebra, $\widetilde{\mathcal{F}} := \{\mathcal{F}_\gamma B := \mathcal{F}_\gamma A \cap B \mid \gamma \in \Gamma\}$ is an induced filtration on B.

Definition 2: $\mathcal{F}_\gamma^* := \sum_{\beta < \gamma} \mathcal{F}_\beta A \quad \forall \gamma \in \Gamma\setminus e$, $G_\gamma := \mathcal{F}_\gamma / \mathcal{F}_\gamma^*$

$G_\alpha A \times G_\beta B \longrightarrow G_{\alpha\circ\beta} A$, $(a + \mathcal{F}_\alpha^* A)\cdot(b + \mathcal{F}_\beta^* A) \mapsto a\cdot b + \mathcal{F}_{\alpha\circ\beta}^* A$

$gr^{\mathcal{F}} A = \bigoplus_{\gamma\in\Gamma} G_\gamma = \bigoplus_{\gamma\in\Gamma} \mathcal{F}_\gamma / \mathcal{F}_\gamma^*$, the assoc. graded algebra.

Note/HW: $gr^{\mathcal{F}}A$ is a Γ-graded algebra

Example 3: $A = K\langle x, \partial \mid \underbrace{\partial x}_{\deg 2} = \underbrace{x\partial}_{2} + \underbrace{1}_{0} \rangle$

(i) $\mathcal{F}_d A := \{ f \in A \mid \deg(f) \leq d \}$, i.e. $\deg x = 1$, $\deg \partial = 1$

observe: $(x^a \partial^b + \text{l.o.t.}) \cdot (x^c \partial^d + \text{l.o.t.}) = x^{a+c} \partial^{b+d} + \text{l.o.t.}$, hence $\mathcal{F}_d A$ is a filtration. $\Gamma = (\mathbb{N}_0, +, 0)$

$\mathcal{F}_0 A = K$, $\mathcal{F}_1 A = K + Kx + K\partial$ $\qquad \partial x + \mathcal{F}_0 A$, $\partial + \mathcal{F}_0 A$

$G_0 A = K$, $G_1 A = K\bar{x} + K\bar{\partial}$

$(\partial + \mathcal{F}_0 A)(x + \mathcal{F}_0 A) = \partial x + \mathcal{F}_1 A = (x\partial + 1) + \mathcal{F}_1 A = x\partial + \mathcal{F}_1 A$

$(x + \mathcal{F}_0 A)(\partial + \mathcal{F}_0 A) = x\partial + \mathcal{F}_1 A$

$=$

(ii) "order filtration": $\mathcal{F}_d = \{ f(x, \partial) \in A \mid \deg_\partial (f(x,\partial)) \leq d \}$, $\deg x = 0$, $\deg \partial = 1$

$\mathcal{F}_0 A = K[x]$ $\quad$ K-basis $\{ x^i \partial^j \mid (i,j) \in \mathbb{N}_0^2 \}$ $\qquad \sim K(x)\langle \partial \mid \cdots \rangle$

$\mathcal{F}_2 A = K[x] + K[x]\partial + K[x]\partial^2$, note that $\dim_K \mathcal{F}_d A = \infty$

$gr^{\mathcal{F}} A \cong K[\bar{x}, \bar{\partial}]$ $\qquad \underbrace{\partial x - x\partial}_{1} = \underbrace{1}_{0}$

(iii) $\mathbb{N}_0^2$-filtration: Fix a monomial well-ordering $<$ on $\mathbb{N}_0^2$ and define $\mathcal{F}_\alpha A := \sum_{\beta \leq \alpha} K x^\beta$. $\qquad x^\alpha < x^\beta \Rightarrow x^{\alpha+\gamma} < x^{\beta+\gamma} \quad \forall \gamma \in \Gamma$

$\mathcal{F}_0 A = K$,

(i) is recovered with $<$ = any degree ordering (deg lex; deg rev lex)

(ii) — " — with $\underset{\text{right lex}}{x < \partial}$, i.e. any monomial with no ∂ in it, is smaller than ∂.

What if $\deg x = -1$, $\deg \partial = 1$? $\qquad \underbrace{\partial x - x\partial}_{0} = \underbrace{1}_{0}$

$\Rightarrow gr^{\mathcal{F}} A \cong A$, $\Gamma = \mathbb{Z}$.

$\mathcal{F}_z A = \sum_{a+b \leq z} K \cdot x^a \partial^b$

Definition 4: For a Γ-filtration $\mathcal{F}$ on A, a left A-module is called Γ-filtered, if $\exists$ K-vsp. $\{ \mathcal{F}_\gamma M \mid \gamma \in \Gamma \}$, $\mathcal{F}_\gamma M \subseteq M$, such that

(i) $M = \sum_{\gamma \in \Gamma} \mathcal{F}_\gamma M$

(ii) $\alpha < \beta \Rightarrow \mathcal{F}_\alpha M \subseteq \mathcal{F}_\beta M$

(iii) $(\mathcal{F}_\alpha A) \cdot (\mathcal{F}_\beta M) \subseteq \mathcal{F}_{\alpha \circ \beta} M \quad \forall \alpha, \beta \in \Gamma$

Let $\mathcal{F}^*_\gamma M := \sum_{\beta < \gamma} \mathcal{F}_\beta$, then $gr^{\mathcal{F}} M := \bigoplus \mathcal{F}_\gamma M / \mathcal{F}^*_\gamma M$ is an <u>assoc. Γ-graded left-module</u> over $gr^{\mathcal{F}} A$.

$$G_\alpha A \times G_\beta M \longrightarrow G_{\alpha \circ \beta} M \ , \ (a + \mathcal{F}^*_\alpha A, m + \mathcal{F}^*_\beta M) \mapsto a \cdot m + \mathcal{F}^*_{\alpha \circ \beta} M .$$

<u>Note</u>: For a submodule $N \subseteq M$, there is an <u>induced Γ-filtration</u>

$$\{ \tilde{\mathcal{F}}_\gamma N := \mathcal{F}_\gamma M \cap N \mid \gamma \in \Gamma \} \ , \quad \{ \mathcal{F}_\gamma (M/N) := \mathcal{F}_\gamma M + N / N \mid \gamma \in \Gamma \}.$$

K-algebras $A \xrightarrow{\psi} B$

Γ-filtrations $\mathcal{F}A$ $\tilde{\mathcal{F}}B$

Γ-filtered modules $M^{\mathcal{F}} \xrightarrow{\psi} N^{\tilde{\mathcal{F}}}$

ψ is <u>$(\Gamma, \mathcal{F}, \tilde{\mathcal{F}})$-filtered</u>
$\Leftrightarrow \psi(\mathcal{F}_\gamma A) \subseteq \tilde{\mathcal{F}}_\gamma B$

ψ is <u>Γ-filtered</u>, if $\psi(\mathcal{F}_\gamma M) \subseteq \tilde{\mathcal{F}}_\gamma N \ \forall \gamma \in \Gamma$

ψ is <u>strict</u>, if $\psi(\mathcal{F}_\gamma M) = \psi(M) \cap \tilde{\mathcal{F}}_\gamma N \ \forall \gamma \in \Gamma$.