(I) Algebra

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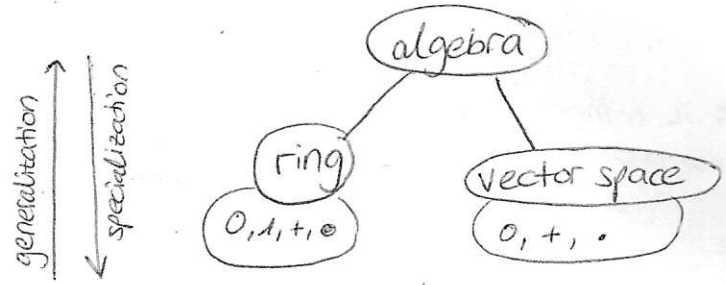

algebra A
- k is a field
- set
- $0, 1 \in A$, $0 \neq 1$
- $+, \circ$ $A \times A \to A$, $k \times A \to A$

Examples: (1) $A = k$ (operations from field ✓).

(2) $\forall A$ algebra: $k \to A$, $\alpha \mapsto \alpha \cdot 1$ linear and injective, so we say "$k \subset A$".

(3) $n \geq 1$: $A = M_n(k)$ with $0 = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}$, $1 = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$,

$+, \circ$ for matrices, $\cdot$ scalar product , $k \to A$, $\alpha \mapsto \begin{pmatrix} \alpha & & \\ & \ddots & \\ & & \alpha \end{pmatrix}$.

(4) V vector space , $A = End(V) = \{ f: V \to V \text{ linear} \}$,

$0 = (v \mapsto 0)$, $1 = Id_V$, $+: (f+g)(v) = f(v) + g(v)$, $\circ$ $(f \circ g)(v) = f(g(v))$, $\cdot$ $(\alpha f)(v) = \alpha f(v)$.

(5) G finite group , $A = k[G] \simeq \bigoplus_{g \in G} \underset{\uparrow \text{basis elements}}{k g} \simeq \bigoplus_{g \in G} k t_g$ (1d-vector space)

i.e. A has a basis indexed by G

... as vector space + multiplication $t_g \circ t_h = t_{gh}$

$\boxed{k \to A, \; \alpha \mapsto \alpha t_e}$

$$\left(\sum_{g \in G} \alpha_g t_g\right) \circ \left(\sum_{g' \in G} \alpha'_{g'} t_{g'}\right) = \sum_{g, g' \in G} \alpha_g \alpha'_{g'} t_{gg'} = \sum_{h \in G} \left(\sum_{gg' = h} \alpha_g \alpha'_{g'}\right) t_h$$

(6) $A = k[X]$, $+, \circ$ addition and multiplication of polynomials, $k \to A$, $\alpha \mapsto \alpha$ (constant polynomial)

(7) $\forall A$ algebra, $I \trianglelefteq A$ (ideal in the ring A), $a \in A$: $aI \subset I$, $Ia \subset I$, $I + I \subset I$

$\Rightarrow \forall \alpha \in k: \alpha \cdot I \subset I \Rightarrow I$ is a subspace

$\Rightarrow A/I$ (the quotient ring) is a vector space

$\Rightarrow A/I$ is an algebra.

(8) $A = k[X]/\langle X^2 \rangle \underset{\text{as algebras}}{\simeq} \dfrac{k \oplus kX \oplus kX^2 \oplus \dots}{kX^2 \oplus kX^3 \oplus \dots} \simeq k \oplus kX = \{ \alpha + \beta X \mid \alpha, \beta \in k \}$.

product: $(\alpha + \beta X) \circ (\alpha' + \beta' X) = \alpha \alpha' + (\alpha \beta' + \alpha' \beta) X$ $\quad \to$ graded (X^2 is homogeneous) — beautiful ♡

$k[X]/\langle X^2 + X + 1 \rangle$ not graded

(9) associative algebras $X \times X \to X,\ (a,b) \mapsto a \cdot b \quad \forall (a,b) \in X \times X \quad a \cdot b \in X$

$\Rightarrow$ X is a magma (or a groupoid)

- an assoc. magma is a semigroup
- a semigroup with a neutral element is a monoid
- monoid, in which every element is invertible $\Rightarrow$ a group

(10) For a unital ring R and a magma X, form the magma ring $(RX, +, \cdot)$ as follows:

- as a (carrier) set $RX = \{ \sum_{i \in I} r_i x_i \mid r_i \in R, x_i \in X,$ finitly many $r_i \neq 0 \}$
- $(\sum_{i \in I} r_i x_i)(\sum_{j \in J} r_j x_j) = \sum_{(i,j) \in I \times J} (r_i r_j)(x_i x_j)$
- addition: component-wise

(11) $k[X,Y] / \langle x-y \rangle \cong k[X]$, $k[X,Y] / \langle x^2, y^5 \rangle$, $k[X,Y] / \langle xy-1 \rangle$

(12) $k\langle X,Y \rangle$ free noncommutative algebra

- vector space with basis $1, x, y, x^2, y^2, xy, yx, x^3, xyx, x^2y, yxy, y^2x, y^3$
  $= \{$words in $x, y\}$
- multiply by concat $(xyx^2) \cdot (x^2yx^3) = xyx^4yx^3$

(13) $k\langle x, y \rangle \longrightarrow M_2(k)$ algebra map

$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\Rightarrow$ $xy \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $yx \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$x^2 \mapsto 0$, $y^2 \mapsto 0$

$xy + yx \mapsto 1$

$\rightsquigarrow$ surjective

$\rightsquigarrow$ $\ker = \langle x^2, y^2, xy + yx - 1 \rangle$

$M_2(k) \cong k\langle x,y \rangle / \langle x^2, y^2, xy + yx - 1 \rangle$

(14) $\mathcal{F} = C^\infty(\mathbb{R}, \mathbb{R}) \ni f(x)$, $X : \mathcal{F} \to \mathcal{F}$, $D : \mathcal{F} \to \mathcal{F}$

$X(f(x)) = x \cdot f(x) \qquad D(f(x)) = \frac{df(x)}{dx}$

$(D \circ X)(f) = \frac{d(xf(x))}{dx} = f(x) + x \frac{df}{dx} = (1 + X \circ D)(f(x)) \Rightarrow D \circ X = X \circ D + 1$

$k\langle X, D \mid DX = XD + 1 \rangle$ 1st polynomial Weyl algebra (named: $A_1, \mathcal{D}_1, \mathcal{D}, \ldots$)

(i) the k-basis of A_1 is $\{ X^a D^b \mid a, b \in \mathbb{N}_0 \}$

(ii) sequences: shift-operator $n \mapsto n+1$

$N : g(n) \mapsto n \cdot g(n)$, $S : g(n) \mapsto g(n+1)$

$SN = (N+1)S = NS + S$

- 1st polynomial shift algebra

Recall: $A \cong K\langle \underline{X}\rangle / I$

$K\langle \underline{X}\rangle$ ← free assoc. algebra in $\underline{X}$

A ← associative algebra over a field K

I ← two-sided ideal of "relations of an algebra"

Def.: A is finitely generated $\iff \exists X'$ s.t. $|X'| < \infty$ and $A \cong K\langle X'\rangle / I$

if $|X| < \infty$, then A is finitely presented [f.p.a.],
if I has finitely many generators.

Ex.: ∞-generated ideal $\langle \{xyx, xy^2x, \dots, xy^ix, \dots\}\rangle$, $X = \{x, y\}$

Gröbner bases [GB] of ideals in $K\langle \underline{X}\rangle$: a canonical basis (generating set)

↳ Only knowing a GB of an ideal I of relations, we can say something about A.

↳ Poincaré-Birkhoff-Witt Theorem for universal enveloping algebras of Lie algebras so-called "Jacobi identity" comes from GB.

Example 1: The q-shift operator; $q \in K \setminus \{0\}$ ("q-dilation")

(recall: to $x \cdot \_$; $\frac{\partial}{\partial x}(-) \rightsquigarrow K\langle x, \partial \mid \partial x = x\cdot\partial + 1\rangle$)

$S_q : f(x) \mapsto f(q\cdot x)$; $X : f(x) \mapsto x \cdot f(x)$

$S_q(X(f(x))) = S_q(x\cdot f(x)) = qx\cdot f(qx) = q \cdot X \cdot S_q(f(x))$

$K\langle X, S_q \mid S_q X = q X S_q\rangle$ $\qquad e^{\hbar}$

1) $q \longrightarrow 1$, $\exists m : q^m = 1$

2) $\forall m : q^m \neq 1$ $\rightsquigarrow$ q is transcendent ~~over~~ K ($\mathbb{Q}$)

Definition 2: $Z(A) = \{a \in A \mid \forall b \in A : ba = ab\}$ the center of A.

$C_A(S) = \{a \in A \mid \forall s \in S : as = sa\}$ the centralisator, for $S \subseteq A$

Facts:
- $Z(A)$ is a K-algebra.
- $C_A(S)$ — " — .

Example 1 [continued]: $q^2 = 1$, $q \neq 1 \Rightarrow q = -1$,

$A := K\langle x, y \mid yx = -xy\rangle$ K-basis $\{x^a y^b \mid a, b \in \mathbb{N}_0\}$

$y \cdot x^n = -xyx^{n-1} = (-1)^n xy$, $y^m x = (-1)^m xy$

Question: - determine the center of A: $Z(A) = K$

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- $C_A(x) = \{a \in A \mid ax = xa\} = K[x, y^2]$: $a = \sum c_{ij} x^i y^j$, $ax = \sum c_{ij} x^{i+1} y^j (-1)^j$.
  $xa = \sum c_{ij} x^{i+1} y^j$

  ~~ad~~: $ax = xa \Rightarrow a = \sum c_{ij} x^i y^{2j}$

- $C_A(y) = K[x^2, y]$

→ 2

Fact: $C_A(x) \cap C_A(y) = Z(A) \Rightarrow K[x,y^2] \cap K[x^2,y] = K[x^2,y^2]$

Observation: A is a finitely generated free module over $K[x^2,y^2] = Z(A)$

$K[x,y] : K[x^2,y^2]$

$$f(x,y) = \sum a_i(x,y) \cdot g_i(x^2,y^2) = 1 \cdot g_1 + x \cdot g_2 + y \cdot g_3 + xy \cdot g_4$$

$(x^3 = x \cdot x^2 \;,\; y^5 = y \cdot (y^2)^2$ basis: $1, x, y, xy$

$K[x,y] : K[x^p, y^p] \rightsquigarrow$ basis : p^2

Example 4: $\varphi : K\langle x, \partial \mid \partial x = x\partial + 1 \rangle \xrightarrow{=A_1(K)} \mathrm{Mat}_n(K) = K^{n \times n}$, $x \mapsto \varphi(x)$, $\partial \mapsto \varphi(\partial)$

a K-algebra homomorphism. $\varphi(\partial)\varphi(x) = \varphi(x)\varphi(\partial) + \mathbb{1}$.

$$\Leftrightarrow \underbrace{\mathrm{Tr}([\varphi(\partial), \varphi(x)])}_{0} \overset{L.A.}{=} \mathrm{Tr}(\mathbb{1}) = n \cdot 1_K$$

$\Rightarrow$ it can only happen when $\mathrm{char}(K) \mid n$.

Also, $Z(A_1(K)) = \begin{cases} K, & \text{if } \mathrm{char}(K) = 0 \\ K[x^p, \partial^p], & \text{if } \mathrm{char}(K) > 0 \end{cases}$

Gradings

$(\Gamma, \circ, \varepsilon)$ is a monoid $(\mathbb{N}_0, \mathbb{Z}, \mathbb{N}_0^n, \mathbb{Z}^n)$

Definition 5: A K-algebra A is called Γ-graded, if $\exists$ K-vector spaces $A_j \subseteq A$, $\gamma \in \Gamma$ such that $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$,

$A_\delta \cdot A_\gamma \subseteq A_{\delta \circ \gamma} \quad \forall \gamma, \delta \in \Gamma$

Example 6: $A = K[x]$, $\Gamma = \mathbb{N}_0 \ni \gamma$, $A_0 = K$, $A_\gamma = K \cdot x^\gamma$

$(k_1 x^{n_1} \cdot k_2 x^{n_2} = (k_1 k_2) x^{n_1 + n_2})$

Lemma 7: (a) A_ε is a sub-K-algebra
(b) $\forall \gamma$: A_γ is $(A_\varepsilon, A_\varepsilon)$-bimodule

For rings R_1, R_2, M is an (R_1, R_2)-bimodule, if M ist a left R_1-module, a right R_2-module $[r_1(m r_2) = (r_1 m) r_2 \;\; \forall (r_1, r_2, m) \in R_1 \times R_2 \times M]$

Example 8: Weyl algebra possesses no $\mathbb{N}_0$-grading.

$\mathbb{Z}$-grading? $\partial x - x\partial = 1 \Rightarrow \begin{array}{l} \deg(\partial) = a \\ \deg(x) = -a \end{array}$, $a \in \mathbb{Z} \setminus \{0\}$

Example 9: $K\langle x, y \mid yx = xy + x + y + 1 \rangle$ cannot be graded.

Proof (of Lemma 7): (a) Observe, that $K \subseteq A_\varepsilon$, and $A_\varepsilon \cdot A_\varepsilon \subseteq A_{\varepsilon \circ \varepsilon} = A_\varepsilon$

(b) $\forall \gamma \in \Gamma$: $A_\varepsilon \cdot A_\gamma \subseteq A_{\varepsilon \circ \gamma} = A_\gamma$; same for $A_\gamma \cdot A_\varepsilon \subseteq A_\gamma$

$(r_\varepsilon \cdot a_\gamma) \cdot r_\varepsilon' = r_\varepsilon (a_\gamma \cdot r_\varepsilon')$ assoc. multiplication of A $\forall r_\varepsilon, r_\varepsilon' \in A_\varepsilon$, $a_\gamma \in A_\gamma$ □

Definition 9: $\forall a \in A \setminus \{0\}$ a Γ-graded algebra, $a = \underbrace{a_\alpha + \ldots + a_\omega}_{\text{finitely many } a_\gamma \in A_\gamma}$

(if Γ is an ordered monoid, then $\alpha > \ldots > \omega$,
a_γ "homogeneous of degree γ" or "graded" of degree γ.)

Multiplication: $\underset{(a_\alpha + \ldots + a_\omega)}{\left(\sum_{\gamma \in I} a_\gamma\right)} \cdot \underset{(b_{\alpha'} + \ldots + b_{\omega'})}{\left(\sum_{\delta \in J} b_\delta\right)} = \sum_{\kappa \in \mathcal{K}} \underbrace{\left(\sum_{\gamma \circ \delta = \kappa} a_\gamma b_\delta\right)}_{A_\kappa} = (a_\alpha b_{\alpha'}) + \ldots + (a_\omega b_{\omega'})$

$= c_{\alpha''} + \ldots + c_{\omega''}$.

Homework: (1) $A = K[X,Y]$, $\deg(X) = 3$, $\deg(Y) = 5$: describe $A_0, A_\gamma, \gamma \in \mathbb{N}$

(2) $A = K[X,Y]$, $\deg(X) = -1$, $\deg(Y) = 1$: describe A_0, A_{+k}, A_{-k}. $\in \mathbb{Z}$

(3) over the Weyl algebra over an abstract K (or rather char $K = 0$)

$\partial \cdot x^n = x^n \cdot \partial + n \cdot x^{n-1}$, more generally

$\forall f \in K[x]$: $\partial \cdot f(x) = f(x) \partial + \frac{\partial f}{\partial x}(x)$

$\partial^m x = x \partial^m + m \partial^{m-1}$ $\forall m, n \in \mathbb{N}$

- prove: $\partial^m x^n = \sum_{k=0}^{\min(m,n)} \frac{m!\,n!}{k!} \frac{x^{n-k} \cdot \partial^{m-k}}{(m-k)!\,(n-k)!}$
- $\theta = x \cdot \partial$, check that $x^n \cdot f(\theta) = f(\theta - n) \cdot x^n$

$\partial^n \cdot f(\theta) = f(\theta + n) \cdot \partial^n$

$\partial^m \cdot x^m = \prod_{i=1}^{m} (\theta + i)$, $x^m \partial^m = \prod_{i=0}^{m-1} (\theta - i)$

(4) compare (2) with considerations about $\mathbb{Z}$-grading
($\deg \partial = 1$, $\deg x = -1$).