steps arrive to \( l' \in F_w \setminus \overline{F_w} \). For \( m' = m - \gamma(l) \) we have 
\[ \psi(m') = \psi(m) = f. \]
By construction: \( m' \in \overline{F_w} \cup M \) with \( m' = 0 \) or \( w' < w \).
We can proceed by \( w > w' > \ldots > \gamma \).
\[ m - m' = \gamma(l) + \gamma(l') + \ldots \in \gamma(l) \in \ker(\psi). \]
Since \( \psi(m') = 0 \Rightarrow m' \notin \gamma(l) \). At each step with \( m'' = 0 \) we terminate at exactly \( \gamma \). Hence \( f = \psi(m) = \psi(m') \).

Proof of Proposition 7(ii):

\[ (\text{gr}^c M)_m = \bigoplus_{s \geq 1 \text{ and } s_1 \leq s_2 \leq \ldots \leq s_r} (\text{gr}^c A)_{s_1} \cdot \sigma(S_i), \]
then \( \text{gr}^c M : m = \bigoplus_{s \geq 1 \text{ and } s_1 \leq s_2 \leq \ldots \leq s_r} (\text{gr}^c A)_{s_1} \cdot \sigma(S_i) + m', \)
\[ s_1 > s_2 > \ldots > s_r \Rightarrow m' \in F_{m+1}, a_s \in F_{s}. A. \]
Suppose \( m' = 0 \) and \( \deg(m') = y' < y \). Then
\[ m' = \bigoplus_{s \geq 1 \text{ and } s_1 \leq s_2 \leq \ldots \leq s_r} (\text{gr}^c A)_{s_1} \cdot \sigma(S_i) + m'' \]
with \( m'' = 0 \) or \( m'' \in F_{m+1}, \gamma > \gamma' > \gamma'' \). ...
We stop after finitely many steps (\( F \) is well-ordered) \( \Rightarrow \) some \( m^* = 0 \).
\[ \Rightarrow m = \bigoplus_{s \geq 1 \text{ and } s_1 \leq s_2 \leq \ldots \leq s_r} (\text{gr}^c A)_{s_1} \cdot \sigma(S_i) = F_{m+1}. \Rightarrow M = \bigoplus_{s \geq 1 \text{ and } s_1 \leq s_2 \leq \ldots \leq s_r} (\text{gr}^c A)_{s_1} \cdot \sigma(S_i) \).

Proof of Theorem 6

(i) \( \Rightarrow \) (ii): \( (\text{gr}^c A) \) is a complex, i.e. \( \text{im}(\text{gr}^c A) \subseteq \ker(\text{gr}^c A) \). We need to prove "\( \exists \) ". Let me \( F_w \setminus F_w \times M \), suppose me \( \ker(\text{gr}^c A) \Rightarrow 0 = \psi(m) = (\text{gr}^c A)[m]. \)

If \( \psi(m) = 0 \Rightarrow m \subseteq \ker(\psi) \Rightarrow \exists L \subseteq L : m = \gamma(l) \in \gamma(L) \subseteq F_w \setminus F_w \times M \)
[by strictness of \( \gamma \)]

If \( \psi(m) \neq 0 \), since \( [\psi(m)] = 0 \exists \gamma' \in \gamma' : \psi(m) \in F_w \).
\[ \psi(m) \in \gamma(l) \in \gamma(l) \rightarrow \exists \gamma'' \in \gamma'' \in \gamma(l) : \psi(m) \in \gamma(l) \Rightarrow m = m' \subseteq \ker(\psi) \in \gamma(l) \subseteq F_w \]
\[ \Rightarrow m = m' + \gamma(l'), l' \in \gamma(l) \text{, then } \]
\[ m = [m] = \bigoplus_{s \geq 1 \text{ and } s_1 \leq s_2 \leq \ldots \leq s_r} (\text{gr}^c A)_{s_1} \cdot \sigma(S_i) = \bigoplus_{s \geq 1 \text{ and } s_1 \leq s_2 \leq \ldots \leq s_r} (\text{gr}^c A)_{s_1} \cdot \sigma(S_i) \Rightarrow m \subseteq \ker(\psi) \]
META THEOREM 8: "Filtered graded transfer"

Let \((X,A,M)\) denotes some property \(X\) of an \(A\)-Module \(M\).
If \(M\) is a \(\Gamma\)-filtered \(A\)-module for \(A\) a \(\Gamma\)-filtered ring with compatible filtrations \(FM, FA\), then

\[(X, gr^X A, gr^X M) \implies (X, A, M)\]

It is valid, in particular, for the following properties:

a) left /right Noetherianity
b) left /right Artinianity
c) simplicity of an algebra \(\text{algebra} \ A\) [simple \(\implies\) the only two-sided ideals are \(0\) and \(A\)]
d) simplicity of a module \(\text{module} \ A\) [simple \(\implies\) the only submodules are \(0\) and \(M\)]
e) semisimplicity of an Artinian algebra \(\iff J(A) = 0\)
f) primeness / semi-primeness / domain props. \(\iff\) \(\text{ring} \ R\) is left/right maximal ideal

\[ \begin{align*}
\text{ring} \ R: & \text{ prime: } \forall a, b \in R, aRb \neq \{0\} \Rightarrow a = 0 \lor b = 0 \\
& [\forall \text{ ideals } A, B \subset R : AB = 0 \Rightarrow A = 0 \lor B = 0 ]
\end{align*} \]

\[ \begin{align*}
\text{semiprime: } & \forall x \in R, xRx + \{0\} \\
& [\forall \text{ ideal } A \subset R \exists \text{ ideal } : A^2 = 0 \Rightarrow A = 0 ]
\end{align*} \]

**Proposition 9**: (i) Let \(\phi: M \to N\) a \(\Gamma\)-filtered homomorphism of \(\Gamma\)-filtered left [right] modules \(M, N\). Then

\(gr^\phi X\) is injective (surjective) \(\iff\) \(\phi\) is injective (surjective)

(ii) Suppose that \(L, N \leq M\) be \(\Gamma\)-filtered w.r.t \(FM\).

Consider the \(\Gamma\)-filtrations \(FN = \{F_n^M \cap N\}^\phi, FL = \{F_n^M \cap L\}^\phi\).

If \(L \leq N\), then \(gr^\phi L \leq gr^\phi N\). Moreover, if \(gr^\phi L = gr^\phi N \Rightarrow L = N\).

**proof**: (i) \(0 \to \ker \phi \to M \to N \to \text{coker} \phi \to 0\), apply thm 6:

\(gr^\phi Y\) is injective \(\iff\) \(\ker (gr^\phi \phi) = 0 \Rightarrow 0 \to gr^\phi M \to gr^\phi N \to \text{coker} (gr^\phi \phi) \to 0\) is exact

\(\Rightarrow 0 \to M \to N \to L \to 0\) is exact with \(\phi\) strict and \(L\) s.t. \(gr^\phi L = \text{coker} (gr^\phi \phi)\)

\(\Rightarrow \phi\) is injective and strict.

(ii) \(L \leq N \Rightarrow 0 \to L \to N\) is exact. Since \(FL, FN\) are induced from \(FM, i\) is strict.

By (i) \(0 \to gr^\phi L \to gr^\phi N\) is exact \(\Rightarrow gr^\phi L \leq gr^\phi N\) and

if \(gr^\phi L = gr^\phi N \Rightarrow gr^\phi i\) is surjective, i.e. bijective \(\Rightarrow i\) is bijective

\(\Rightarrow L = N\). \(\square\)
The first filtration on $K[x_1, \ldots, x_n] = R$

$x_1^{d_1} \cdots x_n^{d_n} = x^d$, $d \in \mathbb{N}^n$. \( \mathbb{N}^n \leftrightarrow \{x^d \mid d \in \mathbb{N}^n\} \) is a $K$-basis of $R$.

**Definition:** Let $<$ be a total ordering on $R$. \( \mathbb{N}^n \times \mathbb{N}^n \)

It is a monomial ordering, if \( x^a < x^b : \iff a < b \) on \( \mathbb{N}^n, 0, + \)

and \( \forall \alpha \in \mathbb{N}^n : x^a + y^a < x^b + y^b \). \( \iff x^a x^d < x^b x^d \)

It is a well ordering, if any subset of \( \mathbb{N}^n, 0, + \) has a min. element w.r.t $<$.

**Examples:**

a) lexicographical ord. \( \alpha <_{\text{lex}} \beta \iff \exists k : \alpha_k = \beta_k, \ldots, \alpha_{k+1} < \beta_{k+1} \)

\[ \implies x^\alpha > y^\beta \iff \text{gcd}(\alpha, \beta) = \text{deg}(x) \text{ and } x^\alpha > y^\beta \]

b) weighted degree. for $w \in R, w = (w_1, \ldots, w_n)$. \( \text{deg}_w x^\alpha = \sum w_i \cdot \alpha_i \)

\[ \alpha = (a_1, \ldots, a_n) \implies \text{deg}_w a = \deg \]

\[ \alpha = (2, 3) \implies x^2 \cdot y^3 \text{ is graded in } w\text{-degree } 6 \]

extension of some ordering $<$ is $(w,<)$:

\[ x^d <_w x^e : \iff \text{deg}_w x^d < \text{deg}_w x^e \text{ or } \text{deg}_w x^d = \text{deg}_w x^e \text{ and } x^d < x^e \]

c) reverse lex. \( \alpha <_{\text{revlex}} \beta \iff \exists k : \alpha_k = \beta_k, \ldots, \alpha_{k-1} < \beta_{k-1} \)

**Popular orderings:** degree-lex, degree-reverse lex

**Remark:** $<$ is global $\iff \alpha = x^e$ is the smallest element w.r.t $<$ on $\mathbb{N}^n$.

$<_w$ is global $\iff w_i \geq 0$, usually $w_i \in \mathbb{N}$.

Let \( w = (w_1, \ldots, w_n) \in \mathbb{N}^n \) be given. define $F_w R = \{ \Sigma Kx^d 1 \leq d <_w y^e \}, y^e \in \mathbb{N}^n$

$\leq$ a $w$-extension of some $<$

\((\mathcal{F}_R)(\mathcal{F}_R) \leq \mathcal{F}_{d + f} R \)

\[ \mathcal{F}_d = \{ f \in K[x_1, \ldots, x_n] \mid \text{deg}_w f < d \}, d \in \mathbb{N}^n \]

$\mathcal{F}_d = K[x]$

\[ \mathcal{F}_e = \{ Kx^3 + Ky^2 : \mathcal{F}_{(3,0)} = y_1 \} \approx \mathcal{F}_{(3,0)} + \mathcal{F}_{(0,0)} \]

$Kx^3 + \mathcal{F}_{(3,0)}$